FORMULAE & KEY POINTS

CLASS 12 MATHEMATICS

CH. 13. PROBABILITY

1. CONDITIONAL PROBABILITY

1.1 DEFINITION

If E and F are two events associated with the same sample space of a random experiment, the conditional probability of the event E given that F has already occurred, is given by

$$P(E|F) = \frac{P(E \cap F)}{P(F)}, \quad P(F) \neq 0 \text{ i.e., } F \neq \emptyset$$

REMARK

If all the outcomes are equally likely, then

$$P(E|F) = \frac{\text{number of ouotcomes favourable to } (E \cap F)}{\text{number of ouotcomesm in F}} = \frac{n(E \cap F)}{n(F)}, \quad n(F) \neq 0$$

EXAMPLE

Consider the experiment of tossing three fair coins. The sample space of the experiment is

S = {HHH, HHT, HTH, THH, HTT, THT, TTH, TTT}

Let E be the event 'at least two heads appear' and F be the event 'at most two heads occur'. Then

E = {HHH, HHT, HTH, THH}, F = { HHT, HTH, THH, HTT, THT, TTH, TTT }

and E ∩ F = {HHT, HTH, THH}

Thus, $n(S) = 8$, $n(E) = 4$, $n(F) = 7$

and $P(E) = \frac{1}{2}$, $P(F) = \frac{7}{8}$ and $P(E \cap F) = \frac{3}{8}$

Then, (i) probability of E given that F has already occured $= P(E|F) = \frac{P(E \cap F)}{P(F)} = \frac{\frac{3}{8}}{\frac{7}{8}} = \frac{3}{7}$

(ii) probability of F given that E has already occured $= P(F|E) = \frac{P(F \cap E)}{P(E)} = \frac{\frac{3}{8}}{\frac{4}{8}} = \frac{3}{4}$

REMARK

(i) Since all the outcomes are equally likely,

$$P(E|F) = \frac{n(E \cap F)}{n(F)} = \frac{3}{7}$$

$$\text{Also, } P(F|E) = \frac{n(F \cap E)}{n(E)} = \frac{3}{4}$$

1.2 PROPERTIES OF CONDITIONAL PROBABILITY

Let A, B, E and F be events of a sample space S of an experiment such that $P(F) \neq 0$, then

(i) $P(S|F) = P(F|F) = 1$

(ii) $P((A \cup B)|F) = P(A|F) + P(B|F) - P((A \cap B)|F)$

In particular, if A and B are disjoint events(i.e. $A \cap B = \emptyset$) then

$$P((A \cup B)|F) = P(A|F) + P(B|F)$$

(iii) $P(E'|F) = 1 - P(E|F)$ OR $P(E'|F) + P(E|F) = 1$

where E' is the complementary event of E i.e. $E' =$ (not E)

2. MULTIPLICATION THEOREM ON PROBABILITY

(i) Multiplication rule of probability for two events

Let E and F be events of a sample space S of an experiment such that $P(E) \neq 0$, $P(F) \neq 0$, then

$$P(E \cap F) = P(E)\,P(F|E) = P(F)\,P(E|F)$$

(ii) Multiplication rule of probability for more than two events

If E, F and G are three events of sample space, we have

$$P(E \cap F \cap G) = P(E)\,P(F|E)\,P(G|(E \cap F)) = P(E)\,P(F|E)\,P(G|EF)$$

Similarly, the multiplication rule of probability can be extended for four or more events

REMARK

For convenience, $P(E \cap F)$ is denoted as $P(EF)$

EXAMPLE

Three cards are drawn successively, without replacement from a pack of 52 well shuffled cards. What is the probability that first two cards are kings and the third card drawn is an ace?

SOLUTION

Let K denote the event that the card drawn is king

and A be the event that the card drawn is an ace.

Then, we have to find P (KKA)

Now $P(K) = \frac{4}{52}$

Also, P (K|K) = probability of drawing second card as a king given that first card drawn is a king.

Now there are 3 kings in $(52 - 1) = 51$ cards.

$\therefore\ P(K|K) = \dfrac{3}{51}$

Lastly, $P(A|KK) =$ probability of drawing third card as an ace given that first and second cards drawn are kings.

Now there are 4 aces in the left 50 cards.

$\therefore\ P(A|KK) = \dfrac{4}{50}$

By multiplication law of probability, we have

$P(KKA) = P(K)P(K|K)P(A|KK) = \dfrac{4}{52} \times \dfrac{3}{51} \times \dfrac{4}{50} = \dfrac{2}{5525}$

VERY IMPORTANT REMARK : Let the problem says "**Three cards are drawn successively, without replacement from a pack of 52 well shuffled cards. What is the probability that two cards are kings and one is an ace?"**

Then the order in which the kings and the ace comne is irrelevant.

Hence, the required probability will be $\dfrac{{}^4C_2 \times {}^4C_1}{{}^{52}C_3} = \dfrac{6}{5525}$

3. INDEPENDENT EVENTS

3.1 Definition 1

Two events E and F associated with a sample space S of an experiment are said to be independent events if the probability of occurrence of one of them is not affected by occurrence of the other.

Mathematically, two events E and F are said to be independent, if

$P(F|E) = P(F)$ provided $P(E) \neq 0$

and $P(E|F) = P(E)$ provided $P(F) \neq 0$

3.2 Definition 2

Two events E and F associated with a sample space S of an experiment are said to be independent events

$$P(E \cap F) = P(E) \,.\, P(F)$$

REMARK

(i) Definition 2 has been obtained by applying multiplication theorem in Definition 1 as follows:

By multiplication rule of probability, we have

$$P(E \cap F) = P(E) \,.\, P(F|E)$$

If E and F are independent, then using Definition 1 we get

$$P(E \cap F) = P(E) \,.\, P(F)$$

(ii) Two events E and F are said to be dependent if they are not independent, i.e. if

$$P(E \cap F) \neq P(E) \,.\, P(F)$$

(iii) **(a)** Sometimes there is a confusion between independent events and mutually exclusive events. Term 'independent' is defined in terms of 'probability of events' whereas mutually exclusive is defined in term of events (subset of sample space).

(b) Moreover, mutually exclusive events never have an outcome common, but independent events, may have common outcome.

(c) In other words, two independent events E and F having nonzero probabilities of occurrence can not be mutually exclusive.

Explanation :

Since E and F have non-zero probabilities $\therefore \Rightarrow P(E) \neq 0,\ P(F) \neq 0$

Since E and F are indepent $\therefore P(E \cap F) = P(E) \times P(F) \neq 0$

$\Rightarrow E \cap F \neq \emptyset$

$\Rightarrow$ E and F are not mutually exclusive.

(d) Conversely, two mutually exclusive events E and F having nonzero probabilities of occurrence can not be independent.

Explanation :

Since E and F have non-zero probabilities $\therefore P(E) \neq 0,\ P(F) \neq 0$

Since E and F are mutually exclusive $\therefore P(E \cap F) = 0 \quad (1)$

Let us assume on the contrary that E and F are independent also

Then $P(E \cap F) = P(E) \times P(F)$

$\therefore$ from (1) $P(E) \times P(F) = 0$

$\Rightarrow P(E) = 0$ or $P(F) = 0$

But this contradicts the given fact that E and F have non − zero probabilities

Hence, our assumption "E and F are independent" is wrong

Thus, E and F are not independent

(iv) Two experiments are said to be independent if for every pair of events E and F, where E is associated with the first experiment and F with the second experiment, the probability of the simultaneous occurrence of the events E and F when the two experiments are performed is the product of P(E) and P(F) calculated separately on the basis of two experiments, i.e.,

$$P(E \cap F) = P(E) \times P(F)$$

(v) Three events A, B and C are said to be mutually independent, if

$P(A \cap B) = P(A)\,P(B)$

$P(A \cap C) = P(A)\,P(C)$

$P(B \cap C) = P(B)\,P(C)$

and $P(A \cap B \cap C) = P(A)\,P(B)\,P(C)$

If at least one of the above is not true for three given events, we say that the events are not independent

(vi) If we are performing a series of trials in the same condition then the outcomes of each trial are independent of the outcomes of every other trial.

EXAMPLES

(a) Let a coin is tossed 3 times. Let A be the event getting tail in the first toss, B be the event getting head in the first toss and C be the event getting tail in the third toss, then the events A, B and C are independent.

(b) If three friends appear for an interview for one post, then the probabilities of their selections are independent.

(c) Let 3 cards are drawn with replacement from a well shuffled deck of 52 cards. Let E be the event "getting a king in the first draw", F be the event "getting a king in the second draw" and G be the event "getting an ace in the third draw", then the events E, F and G are independent.

(v) if the events E and F are independent, then
(a) E' and F' are independent
(b) E' and F are independent,
(c) E and F' are independent

(vi) A and B are two independent events, then

$$P(\text{at least one of A and B}) = 1 - P(A')\,P(B')$$

4. PARTITION OF A SAMPLE SPACE

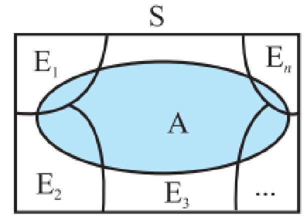
(i) A set of events $E_1, E_2, \ldots, E_n$ is said to represent a partition of the sample space S if
(a) $E_i \cap E_j = \emptyset;\ i \neq j;\ i, j = 1, 2, 3, \ldots, n$
(b) $E_1 \cup E_2 \cup \ldots \cup E_n = S$ and
(c) $P(E_i) > 0$ for all $i = 1, 2, \ldots, n$.
In other words, the events $E_1, E_2, \ldots, E_n$ represent a partition of the sample space S if
(a) they are pairwise disjoint
(b) they exhaustive and
(c) each of them has non-zero probability.

EXAMPLE

In the experiment " simultaneous throw of 3 coins", Sample space, S = { HHH, THH, HTH, HHT, TTH, THT, HTT, TTT}.
Let the events E_1, E_2 and E_3 are defined as follows
E_1 = Getting same outcomes on all the three coins.
E_2 = Getting exactly two heads
E_3 = Getting exactly one Head
Then, the events E_1, E_2 and E_3 form a partition of S.

5. THEOREM OF TOTAL PROBABILITY

(i) Let $\{E_1, E_2, \ldots, E_n\}$ be a partition of the sample space S, and suppose that each of the events $E_1, E_2, \ldots, E_n$ has nonzero probability of occurrence. Let A be any event associated with S, then



$$P(A) = P(E_1)\,P(A|E_1) + P(E_2)\,P(A|E_2) + \ldots + P(E_n)\,P(A|E_n)$$

$$= \sum_{j=1}^{n} P(E_j)\,P(A|E_j)$$

EXAMPLE

A person has undertaken a construction job. The probabilities are 0.65 that there will be strike, 0.80 that the construction job will be completed on time if there is no strike, and 0.32 that the construction job will be completed on time if there is a strike. Determine the probability that the construction job will be completed on time.

SOLUTION

Let A be the event that the construction job will be completed on time,
and B be the event that there will be a strike.
We have to find P(A).
We have
$P(B) = 0.65$, $P(\text{no strike}) = P(B') = 1 - P(B) = 1 - 0.65 = 0.35$
$P(\text{completion of job given that there is a strike}) = P(A|B) = 0.32$,
$P(\text{completion of job given that there is no strike}) = P(A|B') = 0.80$
Since events B and B′ form a partition of the sample space S, therefore, by theorem on total probability, we have
$P(A) = P(B)\,P(A|B) + P(B')\,P(A|B')$
$= 0.65 \times 0.32 + 0.35 \times 0.8$
$= 0.208 + 0.28 = 0.488$
Thus, the probability that the construction job will be completed in time is 0.488

6. BAYES' THEOREM

Let $\{E_1, E_2, \ldots, E_n\}$ be a partition of the sample space S, and Let A be any event of non-zero probability associated with S, then

$$P(E_i|A) = \frac{P(E_i)P(A|E_i)}{\sum_{j=1}^{n} P(E_j)P(A|E_j)}$$

REMARKS

(i) The following terminology is generally used when Bayes' theorem is applied.

(a) The events $E_1, E_2, \ldots, E_n$ are called **Hypotheses.** This is the probability of hypothesis E_i before any new event A is taken into account.

(b) The probability $P(E_i)$ is called the **Priori Probability** of the hypothesis E_i

(c) The conditional probability $P(E_i|A)$ is called a **Posteriori Probability** of the hypothesis E_i. This is the probability of hypothesis E_i after observing event A.

(ii) Bayes' theorem is also called the formula for the **Probability of "Causes"**. Hence, the above formula gives us the probability of a particular E_i (i.e. a "Cause"), given that the event A has occurred.

VERY IMPORTTANT

Many a times students get confused as which event is to be taken as A and which events are to be taken as $E_1, E_2, \ldots, E_n$.

To overcome this confusion, readers should refer to the last part of the question (generally the second last sentence). You will find the part of question which means *"find the probability of the event , say A,* ***given that the event E has already occurred***". Thus, the event A refers to the event A of Bayes' theorem and the event E refer to the event E_i out of the events $E_1, E_2, \ldots, E_n$

EXAMPLE

In a factory which manufactures bolts, machines A, B and C manufacture respectively 25%, 35% and 40% of the bolts. Of their outputs, 5, 4 and 2 percent are respectively defective bolts. A bolt is drawn at random from the product and is found to be defective. What is the probability that it is manufactured by the machine B?

SOLUTION

Let events E_1, E_2, E_3 be the following :

E_1 : the bolt is manufactured by machine A

E_2 : the bolt is manufactured by machine B

E_3 : the bolt is manufactured by machine C

Let the event A be 'the bolt is defective'. $P(E_1) = 25\% = 25/100$, $P(E_2) = 35\% = 35/100$ and $P(E_3) = 40\% = 40/100$

Again, $P(A|E_1)$ = Probability that the bolt drawn is defective given that it is manufactured by machine A = 5% = 5/100

Similarly, $P(A|E_2) = 4\% = 4/100$, $P(A|E_3) = 2\,\% = 2/100$

Hence, by Bayes' Theorem, we have

$$P(E_2|A) = \frac{P(E_2)P(A|E_2)}{P(E_1)P(A|E_1) + P(E_2)P(A|E_2) + P(E_3)P(A|E_3)}$$

$$= \frac{\frac{35}{100} \times \frac{4}{100}}{\frac{25}{100} \times \frac{5}{100} + \frac{35}{100} \times \frac{4}{100} + \frac{40}{100} \times \frac{2}{100}} = \frac{28}{69}$$

REMARK

For ease of calculations, we do not simplify the quantities like 25/100 by writing 1/5, 35/100 writing 7/20 etc. initially. This enables us to cancel all the 100's of the denominators in the second last line of solution.