FORMULAE & KEY POINTS
CLASS 12 MATHEMATICS
CHAPTER 12 /LINEAR PROGRAMMING

VERY IMPORTANT NOTE

The formulation of an LPP is not in CBSE syllabus of Class 12, but readers should understand the formulation to understand the concepts of LPP

1.1 LINEAR PROGRAMMING PROBLEM (LPP)

A linear programming problem is one that is concerned with finding the OPTIMAL VALUE (maximum or minimum) of a linear function of several variables (called OBJECTIVE FUNCTION) subject to the conditions that the variables are non-negative and satisfy a set of linear inequalities (called LINEAR CONSTRAINTS OR OVERRIDING CONDITIONS).
Variables are sometimes called DECISION VARIABLES and are non-negative.

1.2 A few important linear programming problems are:

(i) Diet problems
(ii) Manufacturing problems
(iii) Transportation problems

2. **THE ENTIRE CONCEPT OF THE LINEAR PROGRAMMING WILL BE EXPLAINED THROUGH STEP BY STEP SOLUTION OF THE FOLLOWING REAL LIFE PROBLEM:**

STATEMENT OF THE PROBLEM

A furniture dealer deals in only two items–tables and chairs. He has ₹ 50,000 to invest and has storage space of at most 60 pieces. A table costs ₹ 2500 and a chair ₹ 500. He estimates that from the sale of one table, he can make a profit of ₹ 250 and that from the sale of one chair a profit of ₹ 75. He wants to know how many tables and chairs he should buy from the available money so as to maximise his total profit, assuming that he can sell all the items which he buys.

NOTE

The solution of the above problem has been divided into three parts explaining side by side the new terminologies, concepts and theorems involved.

PART I : Formulation of the LPP

PART II: Representing the LPP on Graph

PART III : Finding the required optimal solution with the help of the graph

2.1 MATHEMTICAL FORMULATION OF A LINEAR PROGRAMMING PROBLEM (LPP)

PART I OF THE SOLUTION : FORMULATION OF LPP

Let x be the number of tables and y be the number of chairs that the dealer buys.
Then obviously, $x \geq 0$ and $y \geq 0$ (1)
These are called **Non-Negative Constraints**

The dealer is constrained by the following:

(i) Maximum amount he can invest is ₹ 50,000
Mathematically, $2{,}500x + 500y \leq 50{,}000$ **(Investment Constraint)** (2)
or $5x + y \leq 100$

(ii) Maximum number of items he can store is 60
Mathematically, $x + y \leq 60$ **(Storage Constraint)** (3)

The dealer's **objective** is to invest in such a way so as to maximise his profit, say, Z.
Thus Z will be a function of x and y mathematically given as
$Z = 250x + 75y$ **(Linear Objective Function)** (4)

Keeping in mind all the things given in the problem, the Linear Programming Problem(LPP) can be stated as:

Maximise $\mathbf{Z = 250x + 75y}$
Subject to the constraints:

$$\mathbf{5x + y \leq 100}$$
$$\mathbf{x + y \leq 60}$$
$$\mathbf{x \geq 0 \text{ and } y \geq 0}$$

2.2 SOME TERMINOLOGIES RELATED TO A LINEAR PROGRAMMING PROBLEM (LPP)

(i) LINEAR OBJECTIVE FUNCTION (OR OBJECTIVE FUNCTION)

The linear function $Z = ax + by$, where a, b are constants, which has to be maximised or minimized is called a linear objective function. Here, the variables x and y are called **DECISION VARIABLES**.

In the above example, $Z = 250x + 75y$ is a linear objective function.

(ii) CONSTRAINTS

The linear inequalities or equations or restrictions on the variables of a linear programming problem are called constraints

In the above example, the set of inequalities (1) to (4) are constraints.

REMARK

In an LPP, the conditions $x \geq 0, y \geq 0$ are called **NON-NEGATIVE CONSTRAINTS**

(iii) OPTIMISATION PROBLEM

A problem which seeks to maximise or minimise a linear function (say of two variables x and y) subject to certain constraints as determined by a set of linear inequalities is called an optimisation problem.

The problem discussed in the above example is an optimization problem as well as an LPP.

REMARK

(i) Linear Programming Problems (LPP) are special type of optimisation problems.

PART II OF THE SOLUTION – REPRESENTING THE LPP ON GRAPH

The LPP as formulated in the above example is:

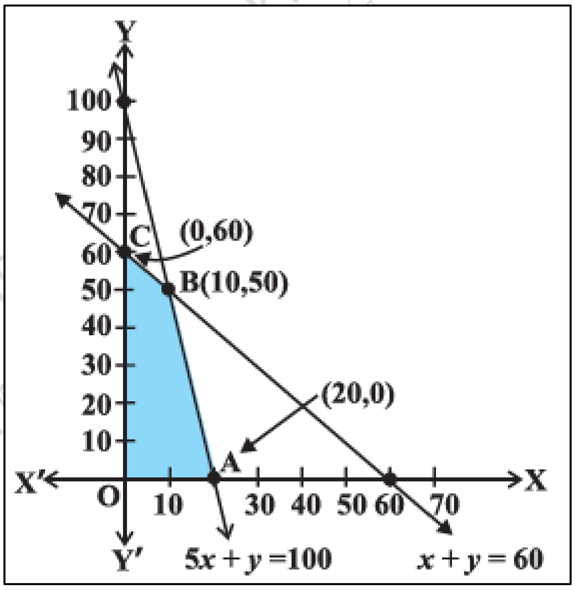
Maximise $Z = 250x + 75y$

Subject to the constraints:

$5x + y \leq 100$ (1)

$x + y \leq 60$ (2)

$x \geq 0$ and $y \geq 0$ (3)



Consider the inequality $5x + y \leq 100$

The corresponding equation is $5x + y = 100$

$x = 0 \Longrightarrow y = 100 \Longrightarrow (0, 100)$ is a solution

$y = 0 \Longrightarrow x = 20 \Longrightarrow (20, 0)$ is a solution

Consider the inequality $x + y \leq 60$

The corresponding equation is $x + y = 60$

$x = 0 \Longrightarrow y = 60 \Longrightarrow (0, 60)$ is a solution

$y = 0 \Longrightarrow x = 20 \Longrightarrow (20, 0)$ is a solution

Putting $x = 0, y = 0$ in (1) we get, $0 \leq 100$ which is true, hence the half plane containing O(0,0) is the solution of the inequality (1)

Putting $x = 0, y = 0$ in (2) we get, $0 \leq 60$ which is true, hence the half plane containing O(0,0) is the solution of the inequality (2)

The shaded area on the graph represents the **feasible region.**

2.3 SOME TERMINOLOGIES RELATED TO THE CORNER POINT METHOD TO SOLVE AN LPP

(i) FEASIBLE REGION (OR SOLUTION REGION) & INFEASIBLE REGION

The common region determined by all the constraints including non-negative constraints $x, y \geq 0$ of a linear programming problem is called the **Feasible Region (Or Solution Region)** for the problem. The region other than feasible region is called an **Infeasible Region.**

In the graph, the region OABC (shaded) is the feasible region for the problem. The region other than feasible region is called an infeasible region.

(ii) FEASIBLE SOLUTIONS & INFEASIBLE SOLUTION

Points within and on the boundary of the feasible region represent **Feasible Solutions** of the constraints.
Any point outside the feasible region is called an **Infeasible Solution** of the constraints.

In the graph, every point within and on the boundary of the feasible region OABC represents feasible solution to the problem. Thus, the points (10, 50), (0, 60), (20, 0) etc. are feasible solutions of the problem whereas the points (25, 40), (0, 100), (60, 0) etc. are infeasible solutions of the problem.

(iii) OPTIMAL (FEASIBLE) SOLUTION:

Any point in the feasible region that gives the optimal value (maximum or minimum) of the objective function is called an optimal solution.

2.4 THEOREMS ON FINDING OPTIMAL SOLUTIONS OF AN LPP

2.4.1 THEOREM 1

In an LPP, an optimal (maximum or minimum) value of the objective function, if it exists, occurs at one or more of the corner points(vertices) of the **convex** feasible region.

2.4.2 THEOREM 2

In an LPP having A **bounded*** feasible region R,

(i) the objective function Z has both a maximum and a minimum value on R

(ii) Each of the maximum and the minimum value of Z occurs at a corner point (vertex) of R.

REMARKS

(i) *A feasible region of a system of linear inequalities is said to be bounded if it can be enclosed within a circle.

(ii) For an LPP having unbounded feasible region R,

(a) The maximum or the minimum value of the objective function Z may or may not exist.

(b) If the maximum or the minimum value of Z exists, it occurs at a corner point of R.

PART III OF THE SOLUTION – FINDING THE NUMBER OF TABLES AND CHAIRS FOR MAXIMISING THE PROFIT

The following table represents the Corner points of the feasible region and the corresponding values of Z at those points

Corner Points	Value of $Z = 250x + 75y$ (in ₹)
O(0,0)	0
C(0, 60)	4500
B (10, 50)	**6250←Maximum**
A (20, 0)	5000

From the above table, in the light of the Theorem 1 and Theorem 2 given above, the dealer should buy 10 tables and 50 chairs to maximise his profit and the maximum profit is ₹ 6250

3. SUMMARY OF THE METHOD TO SOLVE AN LPP BY CORNER POINT METHOD

CASE I : THE FEASIBLE REGION IS BOUNDED

Step 1. Find the feasible region of the linear programming problem and determine its corner points (vertices) either by inspection or by solving the two equations of the lines intersecting at that point.

Step 2. Evaluate the objective function Z = $ax + by$ at each corner point.

Step 3. Let M and m, respectively denote the largest and smallest values of these points. M and m are the maximum and minimum values of Z.

CASE II : THE FEASIBLE REGION IS BOUNDED

Step 1. Find the feasible region of the linear programming problem and determine its corner points (vertices) either by inspection or by solving the two equations of the lines intersecting at that point.

Step 2. Evaluate the objective function Z = $ax + by$ at each corner point.

Now, the following two cases arise

CASE I

To find the maximum value of Z follow these steps

Step 3. Draw the graph of Z > M i.e. $ax + by > \text{M}$

Step 4 (i) If the open half plane determined by Z > M has no point in common with the feasible region then M is the maximum value of Z.

Step 4 (ii) If the open half plane determined by Z > M has points in common with the feasible region then the maximum value of Z does not exist.

CASE II

To find the minimum value of Z follow these steps

Step 3 Draw the graph of Z < m i.e. $ax + by < \text{m}$

Step 4 (i) If the open half plane determined by Z < m has no point in common with the feasible region then m is the minimum value of Z.

Step 4 (ii) If the open half plane determined by Z > m has points in common with the feasible region then the minimum value of Z does not exist.

VERY IMPORTANT REMARK

If for a bounded feasible region, Z has the same maximum (or minimum) value at 2 points, say C and D, then every point on the line segment CD also gives the same maximum (or minimum) value.

THE FOUR MAIN CATEGORIES OF QUESTIONS IN LPP

QUESTION 1.

Solve the following linear programming problem graphically:

Minimise Z = 200 x + 500 y

subject to the constraints:

$x + 2y \geq 10$

$3x + 4y \leq 24$

$x \geq 0, y \geq 0$

SOLUTION

The given function to be minimized is:

$Z = 200x + 500y$

The given constraints are:

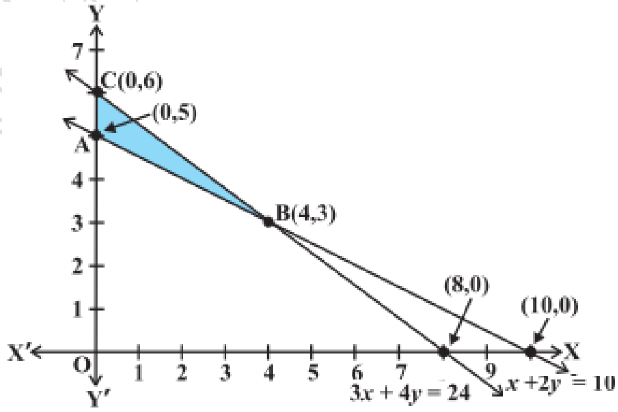
$x + 2y \geq 10$ (1)

$3x + 4y \leq 24$ (2)

$x \geq 0$ and $y \geq 0$ (3)

Consider the inequality $x + 2y \geq 10$
The corresponding equation is $x + 2y = 10$
$x = 0 \Rightarrow y = 5 \Rightarrow (0, 5)$ is a solution
$y = 0 \Rightarrow x = 10 \Rightarrow (10, 0)$ is a solution

Consider the inequality $3x + 4y \leq 24$
The corresponding equation is $3x + 4y = 24$
$x = 0 \Rightarrow y = 6 \Rightarrow (0, 6)$ is a solution
$y = 0 \Rightarrow x = 8 \Rightarrow (8, 0)$ is a solution



Putting $x = 0, y = 0$ in (1) we get, $0 \geq 10$ which is false, hence the half plane not containing O(0,0) is the solution of the inequality (1)
Putting $x = 0, y = 0$ in (2) we get, $0 \leq 24$ which is true, hence the half plane containing O(0,0) is the solution of the inequality (2)

The shaded area on the graph represents the feasible region.

Corner Points	Value of $Z = 200x + 500y$
A(0, 5)	2500
B(4, 3)	2300← Minimum
C (0, 6)	3000

From the above table, the minimum value of Z is 2300 obtained at (4, 3).

QUESTION 2:

Minimize and Maximize $Z = 3x + 9y$ subject to the constraints:

$x + 3y \le 60$

$x + y \ge 10$

$x \le y$

$x \ge 0, \; y \ge 0$

SOLUTION

The Objective Function to be minimized and maximized is

$Z = 3x + 9y$

The given constraints are

$x + 3y \le 60$ (1)

$x + y \ge 10$ (2)

$x \le y$ (3)

$x \ge 0, \; y \ge 0$ (4)

Consider the inequality $x + 3y \le 60$

The corresponding equation is $x + 3y = 60$

$x = 0 \Rightarrow y = 20 \Rightarrow (0, 20)$ is a solution

$y = 0 \Rightarrow x = 60 \Rightarrow (60, 0)$ is a solution

Consider the inequality $x + y \ge 10$

The corresponding equation

is $x + y = 10$

$x = 0 \Rightarrow y = 10 \Rightarrow (0, 10)$ is a solution

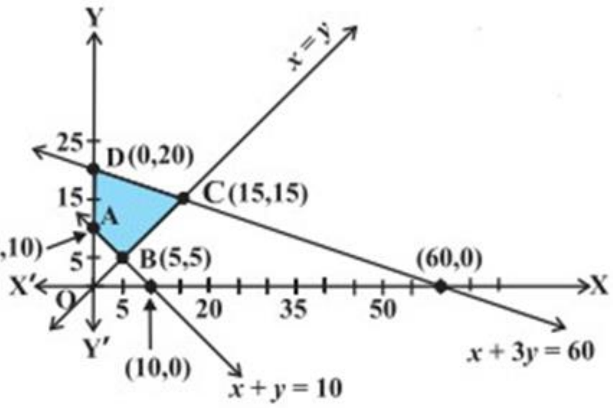
$y = 0 \Rightarrow x = 10 \Rightarrow (10, 0)$ is a solution

Consider the inequality $x \le y$

The corresponding equation is $x = y$

$x = 0 \Rightarrow y = 0 \Rightarrow (0, 0)$ is a solution

$y = 5 \Rightarrow x = 5 \Rightarrow (5, 5)$ is a solution



Putting x = 0, y = 0 in (1) we get, $0 \le 60$ which is true, hence the half plane containing O(0,0) is the solution of the inequality (1)

Putting x = 0, y = 0 in (2) we get, $0 \ge 10$ which is false, hence the half plane not containing O(0,0) is the solution of the inequality (2)

Putting x = 1, y = 0 in (3) we get, $1 \le 0$ which is false, hence the half plane not containing O(0,0) is the solution of the inequality (3)

The shaded area on the graph represents the feasible region.

Corner Points	Value of $Z = 3x + 9y$
A(0,10)	90
B(5, 5)	60 $\leftarrow$ Minimum
C(15, 15)	180 $\leftarrow$ Maximum
D(0, 20)	180 $\leftarrow$ Maximum

From the above table,

The minimum value of Z is 60 obtained at point B(5, 5).

The maximum value of Z is 180 and it occurs at the two corner points C(15, 15) and D(0, 20). Hence the maximum value of Z is 180 and it occurs at every point on the line segment CD

QUESTION 3.

Determine graphically the minimum value of the objective function $Z = -50x + 20y$

subject to the constraints:

$\mathbf{2x - y \geq -5}$

$\mathbf{3x + y \geq 3}$

$\mathbf{2x - 3y \leq 12}$

$\mathbf{x \geq 0, \; y \geq 0}$

SOLUTION 3.

The Objective Function to be minimized is

$Z = -50x + 20y$

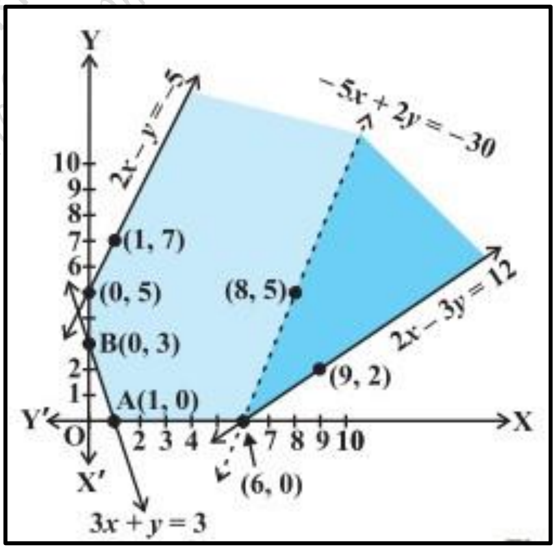
The given constraints are

$2x - y \geq -5$ (1)

$3x + y \geq 3$ (2)

$2x - 3y \leq 12$ (3)

$x \geq 0, \; y \geq 0$ (4)



Consider the inequality $2x - y \geq -5$

The corresponding equation is $2x - y = -5$

$x = 0 \Rightarrow y = 5 \Rightarrow (0, 5)$ is a solution

$x = 2 \Rightarrow y = 9 \Rightarrow (2, 9)$ is a solution

Consider the inequality $3x + y \geq 3$

The corresponding equation is $3x + y = 3$

$x = 0 \Rightarrow y = 3 \Rightarrow (0, 3)$ is a solution

$y = 0 \Rightarrow x = 1 \Rightarrow (1, 0)$ is a solution

Consider the inequality $2x - 3y \leq 12$

The corresponding equation is $2x - 3y = 12$

$y = 0 \Rightarrow x = 6 \Rightarrow (6, 0)$ is a solution

$y = 2 \Rightarrow x = 9 \Rightarrow (9, 2)$ is a solution

Putting x = 0, y = 0 in (1) we get, $0 \geq -5$ which is true, hence the half plane containing O(0,0) is the solution of the inequality (1)

Putting x = 0, y = 0 in (2) we get, $0 \geq 3$ which is false, hence the half plane not containing O(0,0) is the solution of the inequality (2)

Putting x = 1, y = 0 in (3) we get, $0 \leq 12$ which true, hence the half plane containing O(0,0) is the solution of the inequality (3)

The shaded area on the graph represents the feasible region.

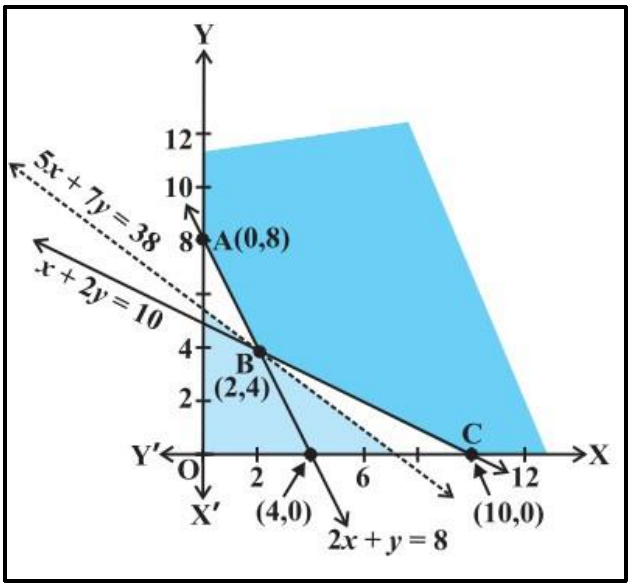
Corner Points	Value of $Z = -50x + 20y$
(0,5)	100
(0, 3)	60
(1, 0)	−50
(6, 0)	−300 ← Minimum

From the above table,
The minimum value of Z is −300. But, the feasible region is unbounded therefore we will consider the inequality
$Z < -300$
$\Rightarrow -50x + 20y < -300$
Dividing both sides by 10 we get
$\Rightarrow -5x + 2y < -30$ (5)
[IMPORTSANT NOTE : If we divide both sides by (−10) to make x-term positive, we will have to reverse the sign of inequality and we will get $5x - 2y > 30$]
The corresponding equation is $-5x + 2y = -30$
$y = 0 \Longrightarrow x = 6 \Longrightarrow (6, 0)$ is a solution
$y = 10 \Longrightarrow x = 10 \Longrightarrow (10, 10)$ is a solution
Putting $x = 0, y = 0$ in (5) we get, $0 < -30$ which is false, hence the half plane not containing O(0,0) and excluding the above line is the solution of the inequality (5)
Since the open half plane represented by (5) has points in common with feasible region, therefore the minimum value of Z does not exist.

QUESTION 4.

Minimise $Z = 50x + 70y$ subject to the constraints:
$2x + y \geq 8$
$x + 2y \geq 10$
$x, y \geq 0$

SOLUTION



The Objective function to be minimised is
$Z = 50x + 70y$
The given constraints are

$2x + y \geq 8$ (1)
$x + 2y \geq 10$ (2)
$x \geq 0, \ y \geq 0$ (3)

Consider the inequality $2x + y \geq 8$ (1)
The corresponding equation is $2x + y = 8$
$x = 0 \Longrightarrow y = 8 \Longrightarrow (0, 8)$ is a solution
$y = 0 \Longrightarrow x = 9 \Longrightarrow (4, 0)$ is a solution

Consider the inequality $x + 2y \geq 10$ (2)
The corresponding equation is $x + 2y = 10$
$x = 0 \Longrightarrow y = 5 \Longrightarrow (0, 5)$ is a solution

$y = 0 \Longrightarrow x = 1 \Longrightarrow (10, 0)$ is a solution

Putting x = 0, y = 0 in the inequality (1) we get, $0 \geq 3$ which is false, hence the half plane not containing O(0,0) is the solution of the inequality (1)
Putting x = 0, y = 0 in the inequality (2) we get, $0 \geq 10$ which is false, hence the half plane not containing O(0,0) is the solution of the inequality (1)

The shaded area on the graph represents the feasible region.

Corner Points	Value of $Z = 50x + 70y$
(0,8)	560
(2, 4)	380 ← Minimum
(10, 0)	500

From the above table,
The minimum value of Z is 380. But, the feasible region is unbounded therefore we will consider the inequality
$Z < 380$
$\Rightarrow 50x + 70y < 380$
$\Rightarrow 5x + 7y < 38$ (4)

The corresponding equation is $5x + 7y = 38 \Longrightarrow x = \dfrac{38 - 7y}{5}$

$y = 4 \Longrightarrow x = 2 \Longrightarrow (2, 4)$ is a solution
$y = 0 \Longrightarrow x = 7.6 \Longrightarrow (7.6, 0)$ is a solution
Putting $x = 0, y = 0$ in (4) we get, $0 < 38$ which is true, hence the open half plane not containing O(0,0) is the solution of the inequality (4)
Since the **open half plane** represented by (4) has no point in common with feasible region, therefore the minimum value of Z is 380 and it occurs at point (2,4).