FORMULAE & KEY POINTS
CLASS 12 MATHEMATICS
CHAPTER 10 : VECTOR ALGEBRA

1. SCALER AND VECTOR QUANTITIES

(i) Scaler Quantity OR Scalers

A quantity which has only **magnitude** and **no direction** is called a scaler quantity.

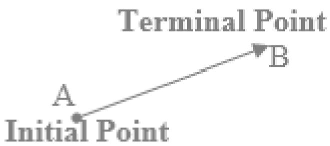
Examples

Scaler Quantities : length, mass, time, distance, speed, area, volume, temperature, work, money, voltage, density, resistance, etc.

Scaler Measures : The measures 5 seconds(time), 1000 cm^3(volume), 30 km/hr(speed), 10 g/cm^3(density), 10 kg(mass), 40 watt(power), 40°(temperature), 10^{-19} coulomb(charge), etc.

(ii) Vector Quantity (OR Vectors)

A quantity which has magnitude as well as direction is called a vector quantity.



Examples

Vector Quantities: **displacement**, **velocity**, **acceleration**, **force**, weight, momentum, electric field intensity, work done, etc.

Vector Measures: 2 meters north-west(velocity), 20 m/s towards north(velocity), 10 Newton(force), 20 m/s^2(acceleration), etc.

REMARKS

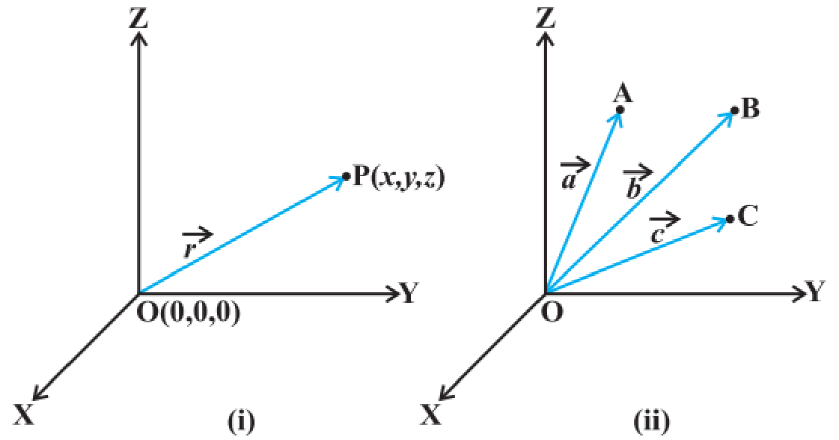
(i) A vector is denoted as $\overrightarrow{AB}$ or $\vec{a}$, and is read as 'vector $\overrightarrow{AB}$' or 'vector $\vec{a}$'.

(ii) The point A from where the vector $\overrightarrow{AB}$ starts is called the '**Initial Point**' and the point B where it terminates (or ends) is called its '**Terminal Point**'.

(iii) The distance between initial and terminal points of a vector is called the magnitude (or length) of the vector, denoted as $|\overrightarrow{AB}|$ or $|\vec{a}|$ or a.

(iv) The arrow – head indicates the direction of the vector.

2. POSITION VECTOR (P.V.) OF A POINT:



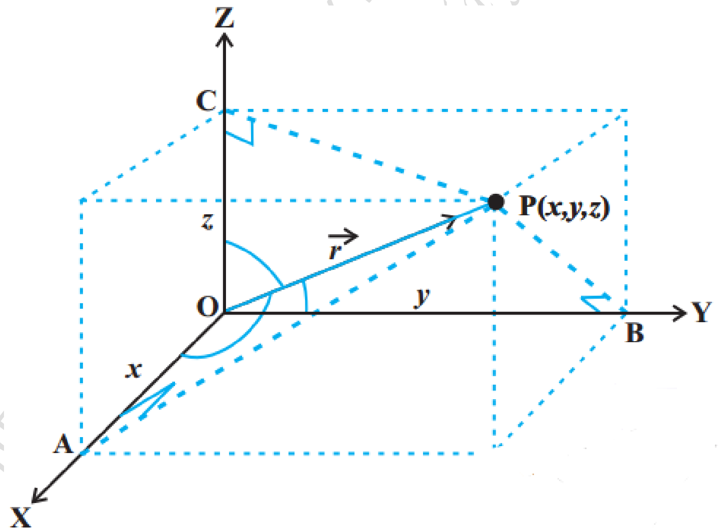
If a point P in space is having coordinates $P(x, y, z)$ with respect to the origin $O(0, 0, 0)$, then, the vector $\overrightarrow{OP} = \vec{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$, having O as its initial point and and P as its terminal point, is called the **Position Vector** of the point P with respect to O.

REMARKS

(i) In practice, the position vector of points A, B, C etc., with respect to origin are denoted respectively $\vec{a}$, $\vec{b}$, $\vec{c}$ etc. respectively

(ii) The magnitude of $\overrightarrow{OP}$ (or $\vec{r}$) is given by $|\overrightarrow{OP}| = \sqrt{x^2 + y^2 + z^2}$

3. DIRECTION ANGLES, DIRECTION COSINES AND DIRECTION RATIOS OF A VECTOR



Let $P(x, y, z)$ be a point in space having position vector $\overrightarrow{OP} = \vec{r}$.
Let $\vec{r}$ makes angles α, β, γ with the positive directions of x, y and z axes respectively. Then,

(A) α, β, γ are called the **Direction Angles** of $\vec{r}$.

(B) $l = \cos\alpha, m = \cos\beta, n = \cos\gamma$ are called the **Direction Cosines (D.C.^{s})** of $\vec{r}$.

(C) Any set of quantities a, b, c proportional to the Direction Cosines l, m, n are called **Direction Ratios (D.R.^{s})** of $\vec{r}$.
Thus, for any non-zero real number k, the Direction Ratios will be

$$a = kl, \qquad b = km, \qquad c = kn$$

REMARKS

(i) $0° \le \alpha, \beta, \gamma \le 180°$

(ii) $l = \cos\alpha, m = \cos\beta, n = \cos\gamma$

$\Rightarrow l = \frac{x}{r}, m = \frac{y}{r}, n = \frac{z}{r} \Rightarrow x = lr, y = mr, z = nr$ where, $r = |\vec{r}|$

$\Rightarrow$ The coordinates of P are $P(lr, mr, nr)$

(iii) $l^2 + m^2 + n^2 = 1.$

(iv) The direction cosines of a vector are **unique** whereas the direction ratios are not so.

(v) The direction cosines of a vector are also its direction ratios.
(because for $k = 1$we get $a = l, b = m$ and $c = n$)

(vi) If a, b, c are the direction ratios of a vector then for any non-zero real number λ, then $\lambda a, \lambda b, \lambda c$ are also the direction ratios of the vector.

(vii) **Relation Between Direction Cosines and Direction Ratios of a Vector**

Let a, b, c be the Direction Ratios of a vector, then the Direction Cosines l, m, n of the vector are either $\frac{a}{\sqrt{a^2+b^2+c^2}}, \frac{b}{\sqrt{a^2+b^2+c^2}}, \frac{c}{\sqrt{a^2+b^2+c^2}}$

or $\frac{-a}{\sqrt{a^2+b^2+c^2}}, \frac{-b}{\sqrt{a^2+b^2+c^2}}, \frac{-c}{\sqrt{a^2+b^2+c^2}}$

4. TYPES OF VECTORS

4.1 ZERO VECTOR:

A vector whose initial and terminal points coincide, is called a zero vector (or null vector), and is denoted as $\vec{0}$.

REMARKS

(i) Zero vector can not be assigned a definite direction as it has zero magnitude.
(ii) Alternatively, a zero vector may be regarded as having any direction.
(iii) The vectors $\overrightarrow{AA}, \overrightarrow{BB}$, etc. also represent zero vector.

4.2 UNIT VECTOR ($\hat{a}$)

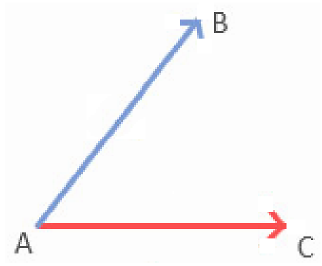
A vector whose magnitude is unity (i. e., 1 unit) is called a unit vector.
The unit vector in the direction of a given vector $\vec{a}$ is denoted by $\hat{a}$.

(i) Unit vector in the direction of $\overrightarrow{AB} = \vec{a}$ is given as $\widehat{AB} = \frac{\overrightarrow{AB}}{|\overrightarrow{AB}|}$ or $\hat{a} = \frac{\vec{a}}{|\vec{a}|}$

(ii) A vector of magnitude k units in the direction of $\vec{a}$ is given as $k(\hat{a}) = \frac{k(\vec{a})}{|\vec{a}|}$

4.3 COINITIAL VECTORS

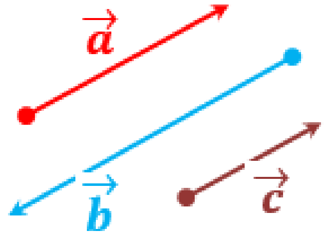
Two or more vectors having the same initial point are called Coinitial Vectors. **Examples** : $\overrightarrow{AB}$ and $\overrightarrow{AC}$



4.4 PARALLEL AND ANTI-PARALLEL VECTORS

Two vectors having same direction are called **Parallel Vectors** whereas two vectors having opposite directions are called **Antiparallel Vectors.**

Example: In the given figure $\vec{a}$ and $\vec{b}$ are parallel vectors whereas $\vec{a}$ and $\vec{c}$; $\vec{b}$ and $\vec{c}$ are anti-parallel vectors.



REMARKS

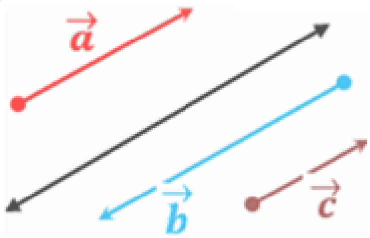
(i) Two vectors $\vec{a}$ and $\vec{b}$ are parallel if $\vec{b} = \lambda\vec{a}$ where **λ is a positive** real number.

(ii) Two vectors $\vec{a}$ and $\vec{b}$ are anti-parallel if $\vec{b} = \lambda\vec{a}$ where **λ is a negative** real number.

4.5 COLLINEAR VECTORS

Two or more vectors are said to be collinear if they are parallel to the same line, irrespective of their magnitudes and directions.

Example: In the given figure $\vec{a}, \vec{b}$ and $\vec{c}$ are collinear vectors.

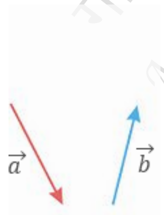


REMARKS

(i) Collinear vectors can be either parallel or anti-parallel.

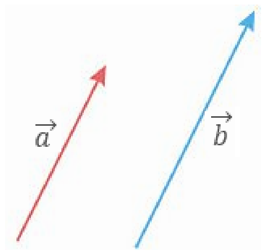
(ii) Two vectors $\vec{a}$ and $\vec{b}$ are collinear if $\vec{b} = \lambda\vec{a}$ where **λ is a non-zero** real number.

4.6 EQUAL VECTORS



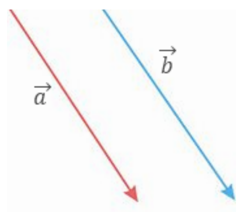
same magnitude, different directions

$\vec{a} \neq \vec{b}$



same direction, different magnitude

$\vec{a} \neq \vec{b}$

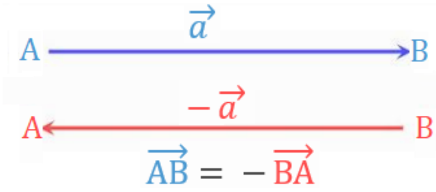


same magnitude as well as direction

$\vec{a} = \vec{b}$

Two vectors $\vec{a}$ and $\vec{b}$ are said to be equal, if they have the same magnitude and same direction regardless of the positions of their initial points, and written as $\vec{a} = \vec{b}$.

4.7 NEGATIVE OF A VECTOR



A vector whose magnitude is the same as that of a given vector (say, $\overrightarrow{AB}$), but direction is opposite to that of it, is called negative of the given vector.

Example: Vector $\overrightarrow{BA}$ is negative of the vector $\overrightarrow{AB}$, and written as $\overrightarrow{BA} = -\overrightarrow{AB}$

4.8 FREE VECTORS

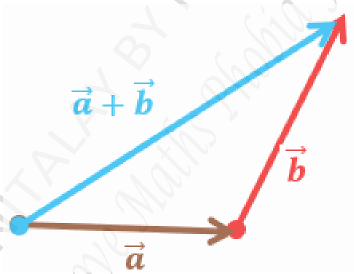
The vectors which may be subject to its parallel displacement without changing its magnitude and direction are called free vectors.

VERY IMPORTANT REMARK

Throughout this chapter, we will be dealing with free vectors only.

5. SUM (OR RESULTANT OR ADDITION) OF VECTORS

5.1 GEOMETRICAL METHOD TO FIND THE SUM (OR RESULTANT) OF TWO VECTORS $\vec{a}$ and $\vec{b}$



STEP I: Place the first vector, say $\vec{a}$ anywhere in a plane.

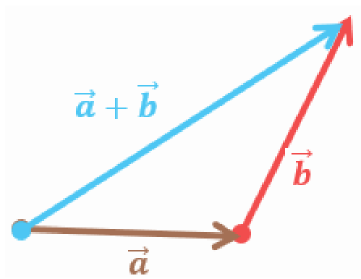
STEP II: Place the second vector, say $\vec{b}$ in the same plane in such a way that the Initial point of $\vec{b}$ coincides with the Terminal Point of $\vec{a}$

STEP III: The sum (or resultant) of the vector $\vec{a}$ and $\vec{b}$, written as $\left(\vec{a} + \vec{b}\right)$ is the vector with initial point same as the initial point of $\vec{a}$ and the terminal point same as the terminal point of $\vec{b}$.

5.2 TRIANGLE LAW OF VECTOR ADDITION

The triangle law for vector addition states that if two vectors are represented in magnitude and direction by two sides of a triangle taken in order then their vector sum is represented by the third side of the triangle taken in the opposite direction.

Example: In the given figure, by triangle law of vector addition $\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$

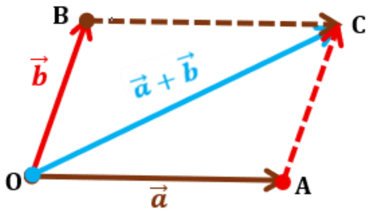


REMARK

In a triangle, if the vectors representing the three sides are taken in order, then their sum is $\vec{0}$. For example, in the above triangle, $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA} = \vec{0}$

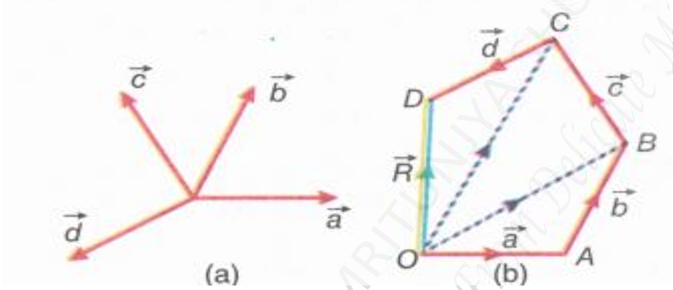
5.3 PARALLELOGRAM LAW OF VECTOR ADDITION

If two vectors $\vec{a}$ and $\vec{b}$ are represented by two adjacent sides of a parallelogram in magnitude and direction, then their sum $\vec{a} + \vec{b}$ is represented in magnitude and direction by the diagonal of the parallelogram through their common initial point.



5.4 POLYGONAL LAW OF VECTOR ADDITION

$$\overrightarrow{OD} = \vec{R} = \vec{a} + \vec{b} + \vec{c} + \vec{d} = \overrightarrow{OA} + \overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD}$$



The polygon law of vector addition states that if a number of vectors are represented in magnitude and direction by the sides of a polygon taken in the same order, their resultant is represented by the closing side of the polygon, taken in the opposite order

6. PROPERTIES OF VECTOR ADDITION

(i) **Commutative Property**: $\vec{a} + \vec{b} = \vec{b} + \vec{a}$

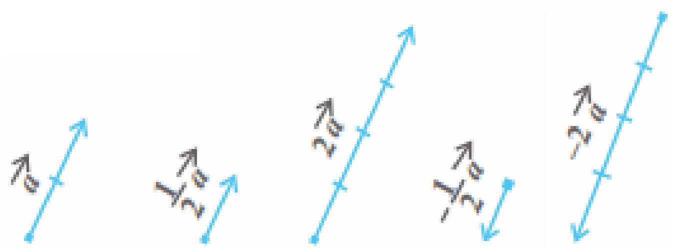
(ii) **Associatative Property**: $(\vec{a} + \vec{b}) + \vec{c} = \vec{a} + (\vec{a} + \vec{b})$

(iii) **Additive Identity**: Zero vector $\vec{0}$ is called the Additive Identity for the vector addition as for any vector $\vec{a}$ we have $\vec{a} + \vec{0} = \vec{0} + \vec{a}$.

(iv) **Additive Inverse**: The vector $-\vec{a}$ is called the negative (or additive inverse) of vector $\vec{a}$ since $\vec{a} + (-\vec{a}) = (-\vec{a}) + \vec{a} = \vec{0}$.

7. MULTIPLICATION OF A VECTOR BY A SCALAR

Let $\vec{a}$ be a given vector and λ a scalar. Then the product of the vector $\vec{a}$ by the scalar λ denoted as $\lambda\vec{a}$, is called the multiplication of vector $\vec{a}$ by the scalar λ.



REMARKS

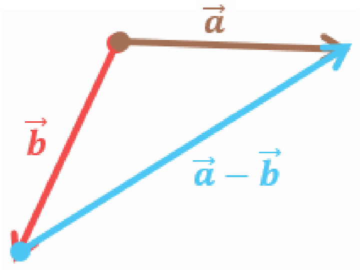
(i) $\lambda \vec{a}$ is a vector, collinear to the vector $\vec{a}$.

(ii) The magnitude of $\lambda \vec{a}$ is $|\lambda|$ times the magnitude of $\vec{a}$ i.e. $|\lambda \vec{a}| = |\lambda||\vec{a}|$

(iii) For any scaler λ, $\lambda \vec{0} = \vec{0}$

8. DIFFERENCE OF TWO VECTORS

GEOMETRICAL METHOD TO FIND THE DIFFERENCE OF TWO VECTORS $\vec{a}$ & $\vec{b}$



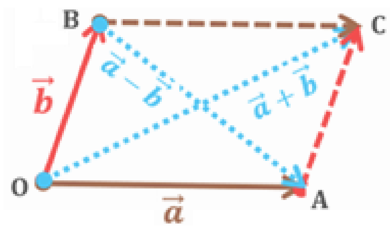
STEP I: Place the first vector, say $\vec{a}$ anywhere in a plane.

STEP II: Place the second vector, say $\vec{b}$ in the same plane in such a way that the Initial point of $\vec{b}$ coincides with the Initial Point of $\vec{a}$

STEP III: The difference of the vector $\vec{a}$ and $\vec{b}$ writen as $(\vec{a} - \vec{b})$is the vector with initial point at the initial point of $\vec{b}$ and the terminal point as the terminal point of $\vec{a}$.

REMARK

If $\vec{a} = \overrightarrow{OA}$ and $\vec{b} = \overrightarrow{OB}$ represent two adjacent sides of a parallelogram OACB, then $(\vec{a} + \vec{b})$ represents the diagonal OC and$(\vec{a} - \vec{b})$ represents the diagonal BA.



9. COMPONENTS OF A VECTOR

Let P(x, y, z) be a point in space in space having position vector $\overrightarrow{OP} = \vec{r}$.

Let $\hat{\imath}, \hat{\jmath}$ and $\hat{k}$ be the unit vectors along OX, OY and OZ respectively, then $\overrightarrow{OP}$ can be written as

$$\overrightarrow{OP} = \vec{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}.$$

This form of any vector is called its **Component Form**.

Here, x, y and z are called as the **Scalar Components** or **Rectangular Components** of $\vec{r}$ and $x\hat{\imath}, y\hat{\jmath}$ and $z\hat{k}$ are called the **Vector Components** of $\vec{r}$ along the respective axes.

REMARK

Length of the Vector, $\vec{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$ is given as

$$|\vec{r}| = |x\hat{\imath} + y\hat{\jmath} + z\hat{k}| = \sqrt{x^2 + y^2 + z^2}$$

10. ALGEBRA OF VECTORS

10.1 If $\vec{a}$ and $\vec{b}$ are any two vectors given in the component form $a_1\hat{\imath} + b_1\hat{\jmath} + c_1\hat{k}$ and $a_2\hat{\imath} + b_2\hat{\jmath} + c_2\hat{k}$ respectively then,

(i) the sum (or resultant) of the vectors $\vec{u}$ and $\vec{v}$ is given by

$\vec{a} + \vec{b} = (\vec{a_1} + \vec{a_2})\hat{\imath} + (\vec{b_1} + \vec{b_2})\hat{\jmath} + (\vec{c_1} + \vec{c_2})\hat{k}$

(ii) the difference of the vectors $\vec{a}$ and $\vec{b}$ is given by

$\vec{a} - \vec{b} = (\vec{a_1} - \vec{a_2})\hat{\imath} + (\vec{b_1} - \vec{b_2})\hat{\jmath} + (\vec{c_1} - \vec{c_2})\hat{k}$

(iii) The vectors $\vec{a}$ and $\vec{b}$ are equal if their corresponding components are equal, that is

$\vec{a_1} = \vec{a_2}, \quad b_1 = b_2, \quad c_1 = c_2$

(iv) The multiplication of vector $\vec{u}$ by any scaler λ is given by

$\lambda\vec{a} = (\lambda a_1)\hat{\imath} + (\lambda b_1)\hat{\jmath} + (\lambda c_1)\hat{k}$

10.2. DISTRIBUTIVE LAWS (OR PROPERTIES)

The addition of vectors and the multiplication of a vector by a scalar together give the following distributive laws:

Let $\vec{a}$ and $\vec{b}$ be any two vectorsc, and k and m be any scalers. Then

(i) $(k + m)\vec{a} = k\vec{a} + m\vec{a}$

(ii) $(km)\vec{a} = k(m\vec{a}) = m(k\vec{a})$

(iii) $k(\vec{a} + \vec{b}) = k\vec{a} + k\vec{b}$

REMARKS

(i) If $\vec{a}$ and $\vec{b}$ are any two vectors given in the component form

$i.e.\ \vec{a} = a_1\hat{\imath} + b_1\hat{\jmath} + c_1\hat{k}$ and $\vec{b} = a_2\hat{\imath} + b_2\hat{\jmath} + c_2\hat{k}$ respectively then,

$\vec{a} \parallel \vec{b} \Leftrightarrow$ there exists a scaler λ such that $\vec{a} = \lambda\vec{b} \Leftrightarrow \frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2} = \lambda$

(ii) The components a_1, b_1, c_1 of $\vec{a}$ are also the Direction Ratios of $\vec{a}$.

(iii) If l, m, n are direction cosines of $\vec{a}$, then $\vec{a} = l\hat{\imath} + m\hat{\jmath} + n\hat{k}$ is the unit vector in the direction of $\vec{a}$

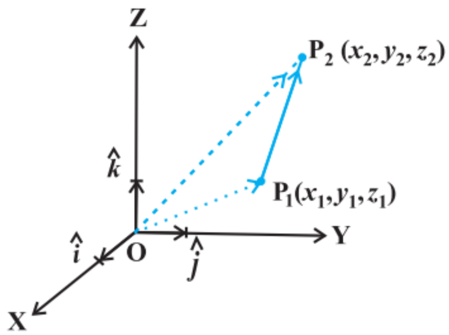
11. VECTOR JOINING TWO POINTS

If A (x_1, y_1, z_1) and B (x_2, y_2, z_2) are any two points, then the vector joining A and B is given as

$\overrightarrow{AB} =$ (Position Vector of B) $-$ (Position Vector of A)

$\Rightarrow \overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$

$\Rightarrow \overrightarrow{AB} = (x_2 - x_1)\hat{\imath} + (y_2 - y_1)\hat{\jmath} + (z_2 - z_1)\hat{k}$



REMARK

The magnitude of $\overrightarrow{AB}$ is given as $|\overrightarrow{AB}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$

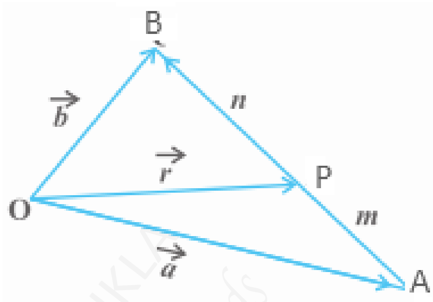
12. SECTION FORMULA

Let A and B be two points with position vectors $\overrightarrow{OA} = \vec{a}$ and $\overrightarrow{OB} = \vec{b}$ respectively. Let P be a point, with position vector $\overrightarrow{OP}$, on the line segment AB. Let the point P divides the line segment joining A and B in the ratio $m : n$. Then the following two cases arise:

12.1 CASE I : Section Formula for Internal Division

If the point P lies between A and B, we say that the point P divides AB internally in the ration $m : n$.
In this case, the position vector of P is given as

$$\overrightarrow{OP} = \vec{r} = \frac{m\,\vec{b} + n\,\vec{a}}{m + n}$$



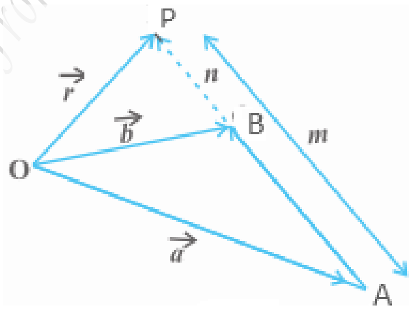
REMARK

The quantities m and n are multiplied with the position vector of the point farther from them.

12.2 CASE II : Section Formula for External Division

If the point P lies on the **extended** line segment AB, we say that the point P divides AB externally in the ration $m : n$.
In this case, the position vector of P is given as

$$\overrightarrow{OP} = \vec{r} = \frac{m\,\vec{b} - n\,\vec{a}}{m - n}$$



REMARK

If $n < m$, the point P lies nearer to the point B and if $n > m$ the point P lies nearer to point A.

12.3 MID – POINT FORMULA

If M is the midpoint of AB, then $m = n$ and the ratio $m : n$ becomes $1 : 1$. Therefore, from Case I, the midpoint M of $\overrightarrow{AB}$,

will have its position vector as $\overrightarrow{OM} = \vec{r} = \dfrac{\vec{a} + \vec{b}}{2}$

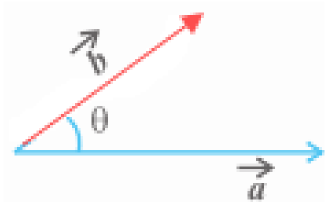
13. PRODUCT OF TWO VECTORS

The product of two vectors can be done in the following two ways:

(**i**) Scalar (or Dot) Product of Two Vectors $(\vec{a}\,.\,\vec{b})$

(**ii**) Vector (or Cross) Product of Two Vectors $(\vec{\boldsymbol{a}} \times \vec{\boldsymbol{b}})$

14. SCALAR (OR DOT) PRODUCT OF TWO VECTORS ($\vec{a}\,.\,\vec{b}$)



The scalar product of two non-zero vectors $\vec{a}$ and $\vec{b}$,

denoted by $\vec{a}\,.\,\vec{b}$ is defined as

$\vec{a}\,.\,\vec{b} = |\vec{a}|\,|\vec{b}|\cos\theta,$

where θ is the angle between $\vec{a}$ and $\vec{b}$, $\quad 0 \le \theta \le \pi$

REMARK

If either $\vec{a} = \vec{0}$ or $\vec{b} = \vec{0}$, then θ is not defined and in this case, we define $\vec{a}\,.\,\vec{b} = 0$

OBSERVATIONS

(i) $\vec{a}\,.\,\vec{b}$ is a real number

(ii) $\vec{a}\,.\,\vec{b} = 0 \Rightarrow \vec{a} = \vec{0}$ **or** $\vec{b} = \vec{0}$ **or** $\vec{a} \perp \vec{b}$

In particular, if $\vec{a}$ and $\vec{b}$ be two non-zero vectors, then $\vec{a}\,.\,\vec{b} = 0 \Leftrightarrow \vec{a} \perp \vec{b}$

(iii) If $\theta = 0$, then $\vec{a}\,.\,\vec{b} = |\vec{a}|\,|\vec{b}|$

In particular, $\vec{a}\,.\,\vec{a} = |\vec{a}|^2$

(iv) If $\theta = \pi$, then $\vec{a}\,.\,\vec{b} = -|\vec{a}|\,|\vec{b}|$

In particular, $\vec{a}\,.\,(-\vec{a}) = -|\vec{a}|^2$

(v) From (ii)and (iii), we get,

$\hat{\imath}.\hat{\imath} = \hat{\jmath}.\hat{\jmath} = \hat{k}.\hat{k} = 1$

$\hat{\imath}.\hat{\jmath} = \hat{\jmath}.\hat{k} = \hat{k}.\hat{\imath} = 0$

(vi) The angle between two nonzero vectors $\vec{a}$ and $\vec{b}$ is given by

$$\cos\theta = \frac{\vec{a}\,.\,\vec{b}}{|\vec{a}|\,|\vec{b}|} \quad \text{or} \quad \theta = \cos^{-1}\left(\frac{\vec{a}\,.\,\vec{b}}{|\vec{a}|\,|\vec{b}|}\right)$$

(vii) The scalar product is commutative. i. e $\vec{a}\,.\,\vec{b} = \vec{b}\,.\,\vec{a}$

(viii) **Distributive Property of Scalar Product Over Addition**:

For any three vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$, we have $\vec{a}\,.\left(\vec{b} + \vec{c}\right) = \vec{a}\,.\,\vec{b} + \vec{a}\,.\,\vec{c}$

(ix) Let $\vec{a}$ and $\vec{b}$ be any two vectors, and λ be any scalar. Then

$\lambda\left(\vec{a}\,.\,\vec{b}\right) = (\lambda\vec{a}).\vec{b} = \vec{a}\,.(\lambda\vec{b})$

IDENTITIES RELATED TO DOT PRODUCT OF VECTORS

In view of the commutative property of dot product of vectors (i.e. $\vec{a}\,.\,\vec{b} = \vec{a}\,.\,\vec{b}$), we have the following identities in vectors:

(i) $(\vec{a}+\vec{b}).(\vec{a}+\vec{b}) = |\vec{a}+\vec{b}|^2 = |\vec{a}|^2 + |\vec{b}|^2 + 2(\vec{a}\,.\,\vec{b})$

(ii) $(\vec{a}-\vec{b}).(\vec{a}-\vec{b}) = |\vec{a}-\vec{b}|^2 = |\vec{a}|^2 + |\vec{b}|^2 - 2(\vec{a}\,.\,\vec{b})$

(iii) $(\vec{a}+\vec{b}).(\vec{a}-\vec{b}) = |\vec{a}|^2 - |\vec{b}|^2$

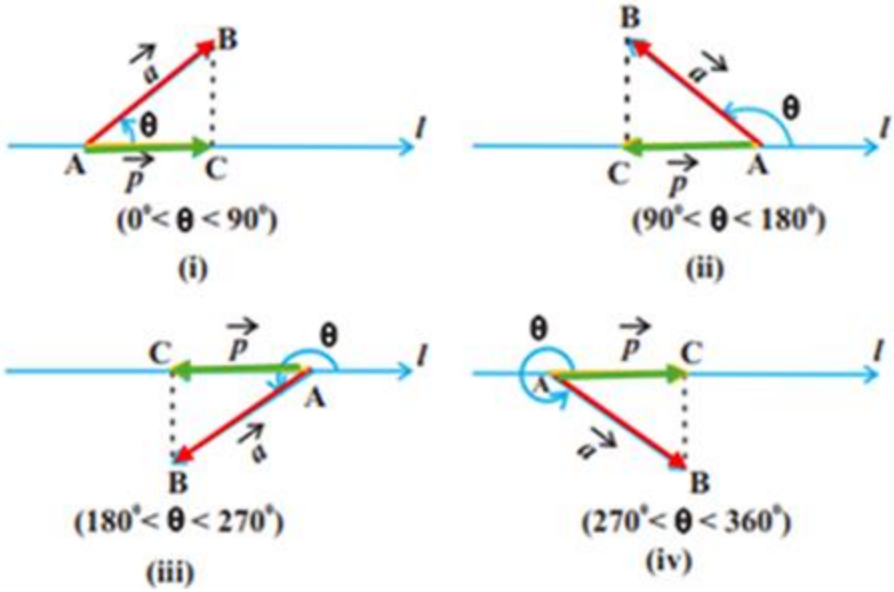
(iv) $(\vec{a}+\vec{b}+\vec{c}).(\vec{a}+\vec{b}+\vec{c}) = |\vec{a}+\vec{b}+\vec{c}|^2$
$= |\vec{a}|^2 + |\vec{b}|^2 + |\vec{c}|^2 + 2[\vec{a}\,.\,\vec{b} + \vec{b}\,.\,\vec{c} + \vec{c}\,.\,\vec{a}]$

(v) $(\vec{a}+\vec{b}-\vec{c}).(\vec{a}+\vec{b}-\vec{c}) = |\vec{a}+\vec{b}-\vec{c}|^2$
$= |\vec{a}|^2 + |\vec{b}|^2 + |\vec{c}|^2 + 2[\vec{a}\,.\,\vec{b} - \vec{b}\,.\,\vec{c} - \vec{c}\,.\,\vec{a}]$

VERY IMPORTANT

The expression $\vec{a}\,\vec{b}$, without any symbol between $\vec{a}$ and $\vec{b}$ is meaningless.
A symbol like ' . ' or ' × ' must be put inbetween $\vec{a}$ and $\vec{b}$ to make the expression meaningful. Similarly, $(\vec{a})^2$ is meaningless.

15. PROJECTION VECTOR($\vec{p}$) AND PROJECTION($|\vec{p}|$) OF A VECTOR ON A DIRECTED LINE l



Suppose a vector $\overrightarrow{AB}$ makes an angle θ with a given directed line l (say), in the anticlockwise direction. Then,

(i) The **Projection Vector** of $\overrightarrow{AB}$ on the directed line l is the vector $\vec{p}$ with magnitude $|\overrightarrow{AB}||\cos\theta|$, and the direction same or opposite to that of the l, depending upon whether $\cos\theta$ is positive or negative.

(ii) The magnitude of $\vec{p}$ i.e. $|\vec{p}| = |\overrightarrow{AB}||\cos\theta|$ is called as the **Projection** of the vector $\overrightarrow{AB}$ on the directed line l.

Example: In each of the above figures (i) to (iv), projection vector of $\overrightarrow{AB}$ along the directed line l is vector $\overrightarrow{AC}$.

OBSERVATIONS

(i) If $\hat{p}$ is the unit vector along a directed line l, then the projection vector of a vector $\vec{a}$ on the line l is given by $\vec{a}.\hat{p}$

(ii) Projection of a vector $\vec{a}$ on other vector $\vec{b}$, is given by

$$\vec{a}.\hat{b} = \vec{a}.\left(\frac{\vec{b}}{|\vec{b}|}\right) = \frac{\vec{a}.\vec{b}}{|\vec{b}|}$$

(iii) If $\theta = 0$, then the projection vector of $\overrightarrow{AB}$ will be $\overrightarrow{AB}$ itself.

(iv) If $\theta = \pi$, then the projection vector of $\overrightarrow{AB}$ will be $\overrightarrow{BA}$ or $-\overrightarrow{AB}$

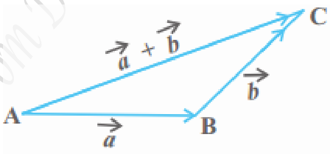
(v) If $\theta = \frac{\pi}{2}$ or $\theta = \frac{3\pi}{2}$, then the projection vector of $\overrightarrow{AB}$ will be zero vector $\vec{0}$.

16. CAUCHY SCHWARTZ INEQUALITY

For any two vectors $\vec{a}$ and $\vec{b}$, $|\vec{a}.\vec{b}| \leq |\vec{a}||\vec{b}|$

17. TRIANGLE INEQUALITY:

For any two vectors $\vec{a}$ and $\vec{b}$, $|\vec{a} + \vec{b}| \leq |\vec{a}| + |\vec{b}|$



REMARK

If equality holds in the Triangle Inequality, that is, $|\overrightarrow{AC}| = |\overrightarrow{AB}| + |\overrightarrow{BC}|$ then the points A, B and C are collinear.

VECTOR (OR CROSS)PRODUCT OF TWO VECTORS $(\vec{a} \times \vec{b})$

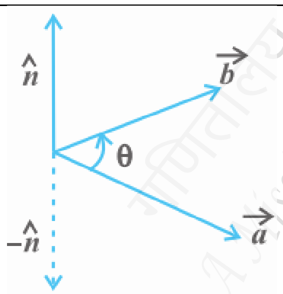
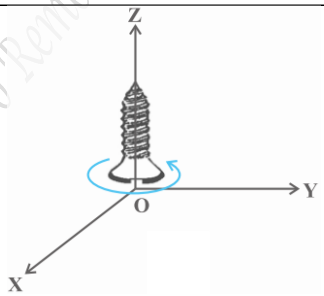
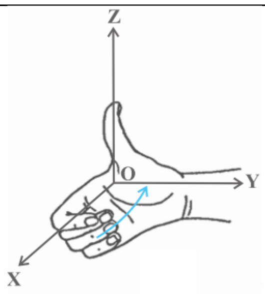
		

Fig. (i) Direction of $\vec{a} \times \vec{b}$	**Fig. (ii)** Right Hand Screw Rule	**Fig. (iii)** Right Hand Thumb Rule

The vector product of two non-zero vectors $\vec{a}$ and $\vec{b}$, denoted by $\vec{a} \times \vec{b}$, is defined as

$$\vec{a} \times \vec{b} = |\vec{a}||\vec{b}| \sin\theta\, \hat{n}$$

where, θ is the angle between $\vec{a}$ and $\vec{b}$ and, $0 \leq \theta \leq \pi$ and

$\hat{n}$ is a unit vector perpendicular to both $\vec{a}$ and $\vec{b}$, such that $\vec{a}, \vec{b}$ and $\hat{n}$ form a right handed system

REMARKS

(i) If $\vec{a}$ and $\vec{b}$ are any two vectors given in the component form $a_1\hat{\imath} + b_1\hat{\jmath} + c_1\hat{k}$ and $a_2\hat{\imath} + b_2\hat{\jmath} + c_2\hat{k}$ respectively then,

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}$$

(ii) If either $\vec{a} = \vec{0}$ or $\vec{b} = \vec{0}$ then θ is not defined and in this case, we define $\vec{a} \times \vec{b} = \vec{0}$

(iii) The direction of the vector $\vec{a} \times \vec{b}$ is obtained by Right Hand Screw Rule [Fig. (ii)] or by Right Hand Thumb Rule [Fig. (iii)].

OBSERVATIONS

(i) $\vec{a} \times \vec{b}$ is a vector quantity.

(ii) $\vec{a} \times \vec{b} = \vec{0} \Rightarrow \vec{a} = \vec{0}$ **or** $\vec{b} = \vec{0}$ **or** $\vec{a}$ and $\vec{b}$ are collinear

In particular, if $\vec{a}$ and $\vec{b}$ be two non-zero vectors, then

$\vec{a} \times \vec{b} = \vec{0} \Leftrightarrow \vec{a}$ and $\vec{b}$ are collinear

(iii) If $\theta = \pi/2$, then $\vec{a} \times \vec{b} = |\vec{a}||\vec{b}|\hat{n}$

(iv) If $\theta = \pi$ or $\theta = 0$, then $\vec{a} \times \vec{b} = \vec{0}$

In particular, $\vec{a} \times \vec{a} = \vec{a} \times (-\vec{a}) = \vec{0}$

VERY IMPORTANT

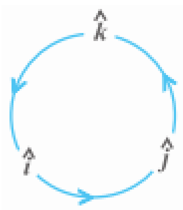
The expressions $\vec{a}\,.\,\vec{b} = \vec{0}$ and $\vec{a} \times \vec{b} = 0$ are meaningles whereas, the expressions $\vec{a}\,.\,\vec{b} = 0$ (a scaler) and $\vec{a} \times \vec{b} = \vec{0}$ (a vector) are meaningful.

(v) From (ii)and (iii), we get,

$\hat{\imath} \times \hat{\imath} = \hat{\jmath} \times \hat{\jmath} = \hat{k} \times \hat{k} = \vec{0}$

$\hat{\imath} \times \hat{\jmath} = \hat{k};\ \hat{\jmath} \times \hat{k} = \hat{\imath};\ \hat{k} \times \hat{\imath} = \hat{\jmath}$

$\hat{\jmath} \times \hat{\imath} = -\hat{k};\ \hat{k} \times \hat{\jmath} = -\hat{\imath};\ \hat{\imath} \times \hat{k} = -\hat{\jmath}$



TRICK TO REMEMBER

Here when the cross product of any two unit vectors out of $\hat{\imath}, \hat{\jmath}, \hat{k}$ are taken in cyclic order as shown in the above figure, then we get the third unit vector in order.
But if the cross product of any two unit vectors out of $\hat{\imath}, \hat{\jmath}, \hat{k}$ are written in reverse order then we get the third vector with '−'sign

(vi) **VERY IMPORTANT**: From the definition of scaler product of two vectors, we get

$$\sin\theta = \frac{|\vec{a} \times \vec{b}|}{|\vec{a}||\vec{b}|} \quad \text{or} \quad \theta = \sin^{-1}\left(\frac{|\vec{a} \times \vec{b}|}{|\vec{a}||\vec{b}|}\right)$$

But we do not use this formula because this formula cannot give whether θ is acute or obtuse as $\sin\theta = \ (\pi - \theta)$. Thus we use the formula $\theta = \cos^{-1}\left(\dfrac{\vec{a}\,.\,\vec{b}}{|\vec{a}|\,|\vec{b}|}\right)$ to find the angle between two vectors.

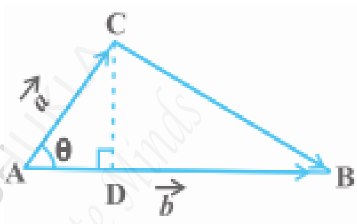
(vii) The vector product is non-commutative. i. e. $\vec{a} \times \vec{b} \neq \vec{b} \times \vec{a}$.
In fact, $\vec{a} \times \vec{b} = -(\vec{b} \times \vec{a})$

(viii) **Distributive Property of Vector Product over Addition:**
For any three vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$, we have $\vec{a} \times \left(\vec{b} + \vec{c}\right) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$

(ix) Let $\vec{a}$ and $\vec{b}$ be any two vectors, and λ be any scalar. Then
$\lambda\left(\vec{a} \times \vec{b}\right) = (\lambda\vec{a}) \times \vec{b} = \vec{a} \times (\lambda\vec{b})$

(x) If $\overrightarrow{AB} = \vec{a}$ and $\overrightarrow{AC} = \vec{b}$ represent the adjacent sides of a triangle then

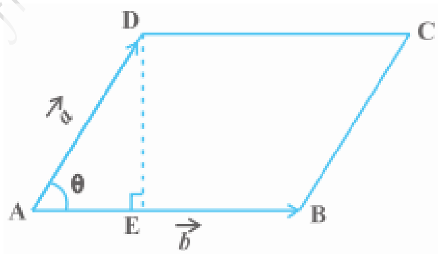
Area of $\Delta ABC = \dfrac{1}{2}|\vec{a}|\,\left|\vec{b}\right| \sin\theta = \dfrac{1}{2}|\vec{a} \times \vec{b}\,|$



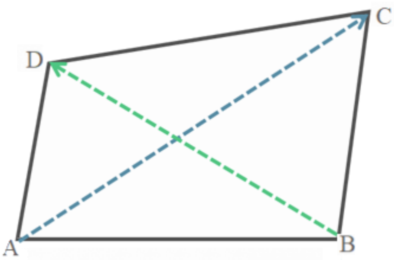
REMARK
For any two non-zero vectors,
The points A, B, C are collinear $\Leftrightarrow$ Area of $\Delta ABC = 0 \Leftrightarrow \vec{a} \times \vec{b} = \vec{0}$
$\Leftrightarrow \vec{a}$ and $\vec{b}$ are collinear

(xi) If $\vec{a}$ and $\vec{b}$ represent the adjacent sides of a parallelogram, then

Area of parallelogram ABCD $= |\vec{a}|\,\left|\vec{b}\right| \sin\theta$
$= |\vec{a} \times \vec{b}\,|$



(xii) The area of the quadrilateral ABCD $= \dfrac{1}{2}\left|\overrightarrow{AC} \times \overrightarrow{BD}\right|$
$= \dfrac{1}{2}$(magnitude of the cross product of diagonals)



(xiii) All the unit vectors in XY -plane are given as $\vec{r} = \cos\theta\ \hat{\imath} + \sin\theta\,\hat{\jmath};\ 0 \le \theta < 2\pi$.