FORMULAE & KEY POINTS

CLASS 12 MATHEMATICS

CHAPTER 09 /DIFFERENTIAL EQUATIONS

1. DIFFERENTIAL EQUATIONS

1.1 DEFINITION OF A DIFFERENTIAL EQUATION

An equation involving derivative (derivatives) of the dependent variable with respect to the independent variable (variables) is called a Differential Equation.

EXAMPLE

$$x\frac{dy}{dx}+y=0$$

1.2 ORDINARY DIFFERENTIAL EQUATION

A differential equation involving **derivatives of the dependent variable with respect to only one independent variable** is called an Ordinary Differential Equation,

EXAMPLE

$$2\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^3=0$$

1.3 PARTIAL DIFFERENTIAL EQUATIONS

Differential equations involving **derivatives with respect to more than one independent variable**, called Partial Differential Equations

VERY IMPORTANT REMARK:

(i) In the syllabus we have only Ordinary Differential Equations.

(ii) Throughout the chapter ‘Differential Equation’ means ‘Ordinary Differential Equation’

1.4 SYMBOLS USED FOR DERIVATIVES:

(i) $\frac{dy}{dx}=y'=y_1=f'(x)=\frac{d}{dx}(f(x))$

(ii) For 1^{st} , 2^{nd} and 3^{rd} order derivatives, we use the following symbols:

$$\frac{dy}{dx} = y'; \quad \frac{d^2y}{dx^2} = y'' = y_2; \quad \frac{d^3y}{dx^3} = y''' = y_3$$

(iii) For convenience, for 4^{th} and higher order derivatives we use the following notations:

$$\frac{d^4y}{dx^4} = y_4; \quad \frac{d^ny}{dx^n} = y_n \text{ etc.}$$

1.5 ORDER AND DEGREE OF A DFFERENTIAL EQUATIONS

(i) ORDER OF A DIFFERENTIAL EQUATIONS

Order of a differential equation is defined as the **order of the highest order derivative** of the dependent variable with respect to the independent variable involved in the given differential equation.

(ii) DEGREE OF A DIFFERENTIAL EQUATIONS

Degree of a differential equation, **when it is a polynomial equation in derivatives**, is the highest power of the highest order derivative involved in the given differential equation.

REMARKS

(i) Order of a differential equation is always defined. But degree of a differential equation may or may not be defined.

(ii) Order and degree (if defined) of a differential equation are always +ve integers (i.e. natural numbers).

(iii) The order of a differential equation representing a family of curves is same as the number of arbitrary constants present in the equation corresponding to the family of curves.

EXAMPLES

(a) The differential equation of the circle $x^2 + y^2 = a^2$ is $\frac{dy}{dx} = -\frac{x}{y}$ which is of order 1.

(b) The differential equation of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is

$$xy\frac{d^2y}{dx^2} + x\left(\frac{dy}{dx}\right)^2 - y\left(\frac{dy}{dx}\right) = 0$$ which is of order 2.

(iv) EXAMPLES FOR ORDER AND DEGREE OF A DIFFERENTIAL EQUATION

	Differential Equation	Order	Degree
(i)	$\left(\frac{dy}{dx}\right)^4 + 3x\frac{d^2y}{dx^2} = 0$	2	1 [as the highest power of y'' is 1]

(ii)	$y = x\dfrac{d^2y}{dx^2} + a\sqrt{1+\left(\dfrac{dy}{dx}\right)^3}$	2	2 [as given eq^n: $(y - xy'')^2 = a^2(1+y')$ So the highest power of $y'' = 2$]
(iii)	$\left(\dfrac{d^2y}{dx^2}\right)^2 + \cos\left(\dfrac{dy}{dx}\right) = 0$	2	Not defined $\left[\text{as } \cos\dfrac{dy}{dx} \text{ is not a polynomial in } \dfrac{dy}{dx}\right]$
(iv)	$((y'')^2 + 2(y')^3 + \sin y = 0$	1	2 [as the power of y'' is 2]

1.6 GENERAL AND PARTICULAR SOLUTIONS OF A DIFFERENTIAL EQUATION

(i) GENERAL SOLUTION OR PRIMITIVE OF A DIFFERENTIAL EQUATION

The solution of a differential equation which contains as many arbitrary constants as the order of the differential equation is called the general solution (or primitive) of the differential equation.

(ii) PARTICULAR SOLUTION OF A DIFFERENTIAL EQUATION

The solution of a differential equation obtained from the general solution by giving particular values to the some or all the arbitrary constants is called a particular solution of the differential equation.

EXAMPLE

For the differential equation $\dfrac{dy}{dx} = -4xy^2$; at $x = 0, y = 1$

The general solution is $y = \dfrac{1}{2x^2 - C}$

After applying the condition at at $x = 0, y = 0$ we get $C = -1$.

$\therefore$ putting $C = -1$ in the general solution we get the particular solution as $y = \dfrac{1}{2x^2+1}$

2. METHOD TO VERIFY THAT A GIVEN FUNCTION IS A SOLUTION OF THE GIVEN DIFFERENTIAL EQUATION

2.1 FOR FUNCTIONS GIVEN IN IMPLICIT FORM

Put the given value of y and its derivatives in the given function to show LHS = RHS

EXAMPLE

Verify that the function $y = a\cos x + b\sin x$, where, $a, b \in R$ is a solution of the differential

equation $\dfrac{d^2y}{dx^2} + y = 0$

SOLUTION

The given function is

$$y = a\cos x + b\sin x \dots (1)$$

Differentiating both sides of equation (1) with respect to x, successively, we get

$$\frac{dy}{dx} = -a\sin x + b\cos x$$

$$\frac{d^2y}{dx^2} = -a\cos x - b\sin x$$

Substituting the values of $\frac{d^2y}{dx^2}$ and y in the given differential equation, we get

$\text{L.H.S.} = (-a\cos x - b\sin x) + (a\cos x + b\sin x) = 0 = \text{R.H.S.}$.

Therefore, the given function is a solution of the given differential equation.

2.2 FOR FUNCTIONS GIVEN IN IMPLICIT FORM

Directly differentiate the given implicit function to show LHS = RHS

OR

By using the given solution and the equation obtained by differentiating the given solution, prove that LHS = RHS.

EXAMPLE

Verify that $x^2 = 2y^2\log y$, is a solution of the differential equation

$$(x^2 + y^2)\frac{dy}{dx} - xy = 0$$

SOLUTION

The given function is

$$x^2 = 2y^2\log y \quad (1)$$

Differentiating both sides of equation (1) with respect to x, we get

$$2x = 2\left(2y\log y\frac{dy}{dx} + y\frac{dy}{dx}\right) \Rightarrow x = \frac{dy}{dx}(2y\log y + y) \qquad (2)$$

From (1), $2y\log y = \frac{x^2}{y}$. Substituting this value in (2)we get

$$x = \frac{dy}{dx}\left(\frac{x^2}{y} + y\right) \Rightarrow x = \frac{dy}{dx}\left(\frac{x^2 + y^2}{y}\right) \Rightarrow xy = \frac{dy}{dx}(x^2 + y^2) \Rightarrow \frac{dy}{dx}(x^2 + y^2) = xy$$

3. TYPES OF FIRST ORDER, FIRST DEGREE DIFFERENTIAL EQUATIONS IN THE SYLLABUS

(i) Differentiable Equations with variables separable.

(ii) Homogeneous Differential Equations

(iii) Linear Differential Equations

➢ VERY IMPORTANT REMARKS

1. If all the terms except C in the general solution are logarithmic functions, then we take constant of integration as log C in place of C(as given in the following example). This enables us to use the property: " $\log m = \log n \Rightarrow m = n$ " to eliminate 'log' on both sides to obtain the solution in simpler form.

2. The following properties of Exponential and Logarithmic functions are also used to obtain the solution in simpler form:

(i) $\log(m \times n) = \log m + \log n$ **(Product Law)**

(ii) $\log\left(\frac{m}{n}\right) = \log m - \log n$ (**Quotient Law**)

(iii) $\log m^n = n \log m$ **(Law of Exponent)**

(iv) (**a**) $\log 1 = 0$ (**b**) $\log e = 1$ (**c**) $\log e^x = x$ (**d**) $e^{\log x} = x$ (**e**) $e^{k \log x} = x^k$

4.1 DIFFERENTIAL EQUATIONS WITH VARIABLES SEPARABLE

The differential equation which can be written in the form

$$\frac{dy}{dx} = f(x) \times g(y)$$

Is called a variable separable differential equation

4.2 METHOD TO SOLVE A DIFFERENTIAL EQUATIONS WITH VARIABLES SEPARABLE

Step 1. Write the equation in the form

$$\frac{dy}{dx} = f(x) \times g(y)$$

Step 2. Separate the variables x and y

$$\frac{1}{g(y)}\,dy = f(x)\,dx$$

Step 3. Integrate both sides with respect to x

$$\int \frac{1}{g(y)}\,dy = \int f(x)\,dx$$

Obtain the general solution of the given equation in the form

$$\mathrm{F}(y) = \mathrm{G}(x) + \mathrm{C}$$

Step 4. If any initial condition is given like, at $x = 0, y = 2$, use it to find the value of C.

Step 5. Substitute the value of C obtained in Step 4. to obtain the particular solution of the given differential equation as

$$\mathrm{F}(y) = \mathrm{G}(x) + k$$

EXAMPLE

Solve the differential equation : $3e^x \tan y \, dx + (1 - e^x) \sec^2 y \; dy = 0$,

given that at $x = 0, y = \frac{\pi}{4}$

(NCERT, CBSE 2011, 2012, 2023)

SOLUTION

Step 1. The given equatioin can be written as

$$\frac{dy}{dx} = \left(\frac{3e^x}{e^x - 1}\right)\left(\frac{\tan y}{\sec^2 y}\right)$$

Step 2. Separate the variables x and y we get

$$\left(\frac{\sec^2 y}{\tan y}\right) dy = \left(\frac{3e^x}{e^x - 1}\right) dx$$

Step 3. Integrate both sides with respect to x we get

$$\int \left(\frac{\sec^2 y}{\tan y}\right) dy = \int \left(\frac{3e^x}{e^x - 1}\right) dx$$

$$\Rightarrow 3 \log(e^x - 1) = \log \tan y + \log \mathrm{C}$$

$$\Rightarrow \log \; (e^x - 1)^3 = \log \; (\mathrm{C} \tan y)$$

$\Rightarrow$ Hence the **general solution** of the given equation is

$$(e^x - 1)^3 = \mathrm{C} \; \tan y$$

Step 4. Putting $x = 1$ and $y = \frac{\pi}{4}$ in the general solution obtained in Step 3, we get

$$(e^1 - 1)^3 \; = \mathrm{C} \; \tan \frac{\pi}{4} \Rightarrow (e - 1)^3 = \mathrm{C} \times 1 \Rightarrow \mathrm{C} = (\mathrm{e} - 1)^3$$

Step 5. Substituting $\mathrm{C} = (\mathrm{e} - 1)^3$ in the general solution obtained in Step 3 we get the particular solution of the given equation as

$$(e^x - 1)^3 = (\mathrm{e} - 1)^3 \; \tan y$$

$$\Rightarrow \; \tan y = \left(\frac{\mathrm{e} - 1}{e^x - 1}\right)^3$$

5. HOMOGENEOUS FUNCTION OF DEGREE n

A function $\mathrm{F}(x, y)$ is said to be homogeneous function of degree n if

$$\mathrm{F}(\lambda x, \lambda y) = \; \lambda^n \, \mathrm{F}(x, y) \quad \text{for any non-zero constant } \lambda$$

OR

A function $\mathrm{F}(x, y)$ is said to be homogeneous function of degree n if

$$\mathrm{F}(x, y) = \; x^n \, g\left(\frac{y}{x}\right) \quad \textbf{or} \quad \mathrm{F}(x, y) = y^n \, h\left(\frac{y}{x}\right)$$

REMARKS

In a homogeneous function, each term of the Numerator has same degree and each term of the denominator has same degree.

EXAMPLE 1

Prove that the function $F(x,y) = y^2 + 2xy$ is a homogeneous function.

SOLUTION

METHOD (1):

Substituting λx for x and λy for y in the given equation we get

$$F(\lambda x, \lambda y) = (\lambda y)^2 + 2(\lambda x)(\lambda y) = \lambda^2(y^2 + 2xy) = \lambda^2 F(x,y)$$

Hence, $F(x,y)$ is a homogeneous function of degree 2

Note that each term of $F(x,y)$ has same degree 2.

METHOD (2): $F(x,y) = y^2 + 2xy = x^2\left[\left(\frac{y}{x}\right)^2 + 2\left(\frac{y}{x}\right)\right] = x^2 g\left(\frac{y}{x}\right).$

Hence, , $F(x,y)$ is a homogeneous function of degree 2

METHOD (3): $F(x,y) = y^2 + 2xy = y^2\left[1 + 2\left(\frac{x}{y}\right)\right] = y^2 h\left(\frac{x}{x}\right).$

Hence, , $F(x,y)$ is a homogeneous function of degree 2

EXAMPLE 2

Prove that the function $F(x,y) = \dfrac{y\cos\left(\frac{y}{x}\right) + x}{x\cos\left(\frac{y}{x}\right)}$ is a homogeneous function.

SOLUTION

Substituting λx for x and λy for y in the given equation we get

$$F(\lambda x, \lambda y) == \frac{\lambda y\cos\left(\frac{\lambda y}{\lambda x}\right) + \lambda x}{x\lambda\cos\left(\frac{\lambda y}{\lambda x}\right)} = \frac{y\cos\left(\frac{y}{x}\right) + x}{x\cos\left(\frac{y}{x}\right)} = \lambda^0 F(x,y)$$

Hence, $F(x,y)$ is a homogeneous function of degree 0.

Note that in $F(x,y)$, each term of the numerator has same degree 1 and each term of the denominator has same degree 1.

EXAMPLE 3

Prove that the function $F(x,y) = \dfrac{3x^2 + 2y^2}{2x + 3y}$ is a homogeneous function.

SOLUTION

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$$F(x,y) = \frac{3x^2 + 2y^2}{2x + 3y}$$

Substituting λx for x and λy for y we get

$$F(\lambda x, \lambda y) = \frac{3(\lambda x)^2 + 2(\lambda y)^2}{2\lambda x + 3\lambda y} = \frac{\lambda^2(3x^2 + 2y^2)}{\lambda(2x + 3y)} = \lambda\left(\frac{3x^2 + 2y^2}{2x + 3y}\right) = \lambda^1 F(x,y)$$

Hence, $\mathrm{F}(x, y)$ is a homogeneous function of degree 1.

Note that in $\mathrm{F}(x, y)$, each term of the numerator has same degree 2 and each term of the denominator has same degree 1.

6.1 HOMOGENEOUS DIFFERENTIAL EQUATION

A differential equation of the form $\frac{dy}{dx} = \mathrm{F}\,(x, y)$ is said to be homogenous differential equation if $\mathrm{F}(x, y)$ is a homogenous function of degree zero.

VERY IMPORTANT REMARKS

(i) A homogeneous differential equation can be written in any one or both of the following two forms,

FORM I : $\frac{dy}{dx} = g\left(\frac{y}{x}\right)$

FORM II : $\frac{dx}{dy} = h\left(\frac{x}{y}\right)$.

(ii) A homogeneous equation of the type $\frac{\mathrm{dy}}{\mathrm{dx}} = \frac{p(x, y)}{q(x, y)}$, where $p(x, y)$ and $q(x, y)$ are polynomials or radical functions in x and y can be written in both the forms given above.

Example:

Consider the homogeneous differential equation $(x^3 - 3xy^2)dx = (y^3 - 3x^2y)dy$ (1)

From (1) $\frac{dy}{dx} = \frac{x^3 - 3xy^2}{y^3 - 3x^2y} \Rightarrow \frac{dy}{dx} = \frac{\left(\frac{y}{x}\right)^3 - 3\left(\frac{y}{x}\right)}{1 - 3\left(\frac{y}{x}\right)^2} \Rightarrow \frac{dy}{dx} = g\left(\frac{y}{x}\right)$ which is FORM I

Also from (1) $\frac{dx}{dy} = \frac{y^3 - 3x^2y}{x^3 - 3xy^2} \Rightarrow \frac{dx}{dy} = \frac{\left(\frac{x}{y}\right)^3 - 3\left(\frac{x}{y}\right)}{1 - 3\left(\frac{x}{y}\right)^2} \Rightarrow \frac{dx}{dy} = h\left(\frac{x}{y}\right)$ which is FORM II

(iii) As per the form of the homogeneous differential equation there are two methods to solve the equation.

6.2 METHODS TO SOLVE A HOMOGENEOUS DIFFERENTIAL EQUATION

(i) METHOD TO SOLVE A HOMOGENEOUS DIFFERENTIAL EQUATION OF THE FORM I

$$\frac{dy}{dx} = g\left(\frac{y}{x}\right)$$

Step 1. Write the given equation the Form I, that is,

$$\frac{dy}{dx} = g\left(\frac{y}{x}\right)$$

Step 2. Let $y = vx \Rightarrow v = \frac{y}{x}$ and $\frac{dy}{dx} = v + x\frac{dv}{dx}$.

Substitute these values in the equation obtained in Step1. (**This substitution transforms the given homogenous differential equation to a variable separable form.)**

Step 3. Reduce the equation obtained in Step 2 into the following variable separable form

$$w(v)\,dv = u(x)\,dx$$

Step 4. Integrate both sides with respect to x

$$\int w(v)\,dv = \int u(x)\,dx$$

Obtain the general solution of the given equation in the form

$$\mathrm{W(v) = U}(x) + \mathrm{C}$$

Step 5. Replace v by $\frac{y}{x}$ in the solution obtained in Step 3 to get the general solution of the given equation

Step 6. If any initial condition is given like, at $x = x_1, y = y_1$, use it to find the value of C.

Step 7. Substitute the value of C obtained in Step 4. to obtain the particular solution of the given differential equation.

EXAMPLE

Show that the differential equation $\frac{dy}{dx} - \frac{y}{x} + \operatorname{cosec}\frac{y}{x} = 0$ is homogeneous and find its particular solution, given that at $y(1) = 0$.

NCERT, CBSE 2009

SOLUTION

Step 1. The given differential equation can be written as

$$\frac{dy}{dx} = \frac{y}{x} - \operatorname{cosec}\frac{y}{x} \qquad (1)$$

Part I : Proving that the given equation is a homogeneous differential equation.

$$\text{Let } \mathrm{F}(x, y) = \frac{y}{x} - \operatorname{cosec}\frac{y}{x}$$

Then $\mathrm{F}(\lambda x, \lambda y) = \frac{\lambda y}{\lambda x} - \operatorname{cosec}\frac{\lambda y}{\lambda x} = \frac{y}{x} - \operatorname{cosec}\frac{y}{x} = \lambda^0\,\mathrm{F}(x, y)$

Thus, $\mathrm{F}(x, y)$ is a homogeneous function of degree zero. Hence the given differential equation is a homogeneous differential equation.

Part II : Solving the homogeneous differential equation

Step 2. Let $y = vx \Rightarrow v = \frac{y}{x}$ and $\frac{dy}{dx}$

$= v + x\frac{dv}{dx}$. Substituting these in the equation (1) we get

$$v + x\frac{dv}{dx} = v - \operatorname{cosec} v$$

TRICK: To obtain the RHS of this equation replace y by v and x by 1 in the RHS of equation (1)

$$\Rightarrow\ x\frac{dv}{dx} = -\operatorname{cosec} v$$

$$\Rightarrow \sin v\, dv = -\frac{1}{x}\, dx$$

Step 3. Integrating both sides with respect to y we get

$$\int \sin v\, dv = -\int \frac{1}{x} dx$$

$$\Rightarrow -\cos v = -\log|x| + C$$

Step 4. Replacing v by $\frac{y}{x}$ we get

$$-\cos\frac{y}{x} = -\log|x| + C \qquad (2)$$

This is the general solution of the given equation

Step 5. Now, given that $y(1) = 0$, substituting $x = 1$ and $y = 0$ in the general solution (2) we get

$$-\cos 0 = -\log 1 + C$$
$$\Rightarrow C = -1$$

Step 6. Substituting $C = -1$ in the general solution (2) we get the particular solution of the given equation as

$$\cos\frac{y}{x} = \log|x| + 1$$

(ii) METHOD TO SOLVE A HOMOGENEOUS DIFFERENTIAL EQUATION OF THE FORM II

$$\frac{dx}{dy} = h\left(\frac{x}{y}\right)$$

Step 1. Write the equation in the Form II that is,

$$\frac{dx}{dy} = h\left(\frac{x}{y}\right)$$

Step 2. Let $x = vy \ \Rightarrow\ v = \frac{x}{y}$ and $\frac{dx}{dy} = v + y\frac{dv}{dy}$. Substite these values in the equation obtained in Step I. (**Note that the above substitution transforms the homogenous differential equation to a variable separable form.)**

Step 3. Reduce the equation obtained in Step 2 into variable separable form

$$w(v)\, dv = u(y)\, dy$$

Step 4. Integrate both sides with respect to y

$$\int w(v)\, dv = \int u(y)\, dy$$

Obtain the general solution of the given equation in the form

$$\mathrm{W(v)} = \mathrm{U}(y) + \mathrm{C}$$

Step 5. Replacing v by $\frac{x}{y}$ in the solution obtained in Step 3 to get the general solution of the given equation

Step 6. If any initial condition is given like, at $x = 0, y = 2$, use it to find the value of C.

Step 7. Substitute the value of C obtained in Step 4. to obtain the particular solution of the given differential equation.

EXAMPLE

Show that the differential equation $2y\, e^{\frac{x}{y}} dx + \left(y - 2x\, e^{\frac{x}{y}}\right) dy = 0$ is homogeneous and find its particular solution, given that at $x = 0, y = 1$.

NCERT, CBSE 2012, 13

SOLUTION

Step 1. The given differential equation can be written as

$$\frac{dx}{dy} = \frac{2x\, e^{\frac{x}{y}} - y}{2y\, e^{\frac{x}{y}}} \qquad (1)$$

Part I : Proving that the given equation is a homogeneous differential equation.

Let $$\mathrm{F}(x, y) = \frac{2x\, e^{\frac{x}{y}} - y}{2y\, e^{\frac{x}{y}}}$$

Then $$\mathrm{F}(\lambda x, \lambda y) = \frac{2\lambda x\, e^{\frac{\lambda x}{\lambda y}} - \lambda y}{2\lambda y\, e^{\frac{\lambda x}{\lambda y}}} = \frac{\lambda\left(2x\, e^{\frac{x}{y}} - y\right)}{\lambda\left(2y\, e^{\frac{x}{y}}\right)} = \lambda^0\, \mathrm{F}(x, y)$$

Thus, $\mathrm{F}(x, y)$ is a homogeneous function of degree zero. Hence the given differential equation is a homogeneous differential equation.

Part II : Solving the homogeneous differential equation

Step 2. Put $x = vy \;\Rightarrow\; v = \frac{x}{y}$ and $\frac{dx}{dy} = v + y\frac{dv}{dy}$. Substituting these values in the equation (1) we get

$$v + y\frac{dv}{dy} = \frac{2v\, e^{v} - 1}{2\, e^{v}}$$

TRICK: To obtain the RHS of this equation replace x by v and y by 1 in the RHS of equation (1)

Step 3. $$\Rightarrow y\frac{dv}{dy} = \frac{2v\, e^{v} - 1}{2\, e^{v}} - v$$

$$\Rightarrow y\frac{dv}{dy} = -\frac{1}{2\,e^{v}} -$$

$$\Rightarrow 2\,e^{v}dv = -\frac{dy}{y}$$

Step 4. Integrating both sides with respect to y we get

$$2\int e^{v}dv = -\int \frac{dy}{y}$$

$$\Rightarrow 2\,e^{v} = -\log|y| + \mathrm{C}$$

Step 5. Replace v by $\frac{x}{y}$ we get

$$2\,e^{\frac{x}{y}} + \log|y| = \mathrm{C} \qquad (2)$$

This is the general solution of the given equation

Step 6. Now substituting $x = 0$ and $y = 1$ in the general solution (2) we get

$$2\,e^{0} + \log 1 = \mathrm{C}$$

$$\Rightarrow \mathrm{C} = 2$$

Step 7. Substituting $\mathrm{C} = 2$ in the general solution (2) we get the particular solution of the given equation as

$$2\,e^{\frac{x}{y}} + \log|y| = 2$$

7.1. FIRST ORDER LINEAR DIFFERENTIAL EQUATION

A differential equation of any of the two following forms is called a First Order Linear Differential Equation:

FORM I

$\frac{dy}{dx} + \mathrm{P}(x) \times y = \mathrm{Q}(x)$ where $\mathrm{P}(x)$ and $\mathrm{Q}(x)$ are functions of x only or constants.

FORM II

$\frac{dx}{dy} + \mathrm{P}(y) \times x = \mathrm{Q}(y)$ where $\mathrm{P}(y)$ and $\mathrm{Q}(y)$ are functions of y only or constants.

7.2 METHODS TO SOLVE A FIRST ORDER LINEAR DIFFERENTIAL EQUATION

7.2.1 METHOD TO SOLVE A FIRST ORDER LINEAR DIFFERENTIAL EQUATION OF THE FORM I

$$\frac{dy}{dx} + \mathrm{P}(x) \times y = \mathrm{Q}(x)$$

STEP 1 . Write the given differential equation in the form $\frac{dy}{dx} + \mathrm{P}(x) \times y = \mathrm{Q}(x)$

STEP 2 . Obtain the values of $\mathrm{P}(x)$ and $\mathrm{Q}(x)$ from the equation of Step 1.

STEP 3 . Find the integrating Factor $(\mathrm{I.F.}) = e^{\int \mathrm{P}(x)dx}$

STEP 4 . Write the solution of the given differential equation as

$$y \times (\text{I. F.}) = \int [\text{Q}(x) \times (\text{I. F.})]\, dx + \text{C}$$

STEP 4 . If any initial condition is given like, at $x = 0, y = 2$, use it to find the value of C.

Step 5. Substitute the value of C obtained in Step 4. to obtain the particular solution of the given differential equation.

EXAMPLE

Solve the initial value problem : $\cos^2 x \frac{dy}{dx} + y = \tan x$; $y(1) = 0$

NCERT, CBSE 2008, 2011, 2023

SOLUTION

STEP 1 . The given differential equation can be written as

$$\frac{dy}{dx} + (\sec^2 x)y = (\tan x)(\sec^2 x)$$

STEP 2 . Here, $\text{P}(x) = \sec^2 x$ and $\text{Q}(x) = (\tan x)(\sec^2 x)$

STEP 3 . Integrating Factor $(\text{I. F.}) = e^{\int \text{P(x)}dx} = e^{\int \sec^2 x\, dx} = e^{\tan x}$

STEP 4 . The solution of the given differential equation is given as

$$y \times (\text{I. F.}) = \int [\text{Q}(x) \times (\text{I. F.})]\, dx + \text{C}$$

$$\Rightarrow y \times (e^{\tan x}) = \int [(\tan x)(\sec^2 x) \times (e^{\tan x})]\, dx + \text{C}$$

Putting $\tan x = t$ and $\sec^2 x\, dx = dt$ on RHS we get

$$y\, e^{\tan x} = \int t\, e^t dt + \text{C}$$

Integrating the integral by parts taking t as first function and e^t as second function we get

$$y\, e^{\tan x} = t \int e^t\, dt - \int \left[\frac{d}{dt}(t) \int \text{e}^t dt\right] dt + \text{C}$$

$$\Rightarrow y\, e^{\tan x} = te^t - \int e^t dt + \text{C}$$

$$\Rightarrow y\, e^{\tan x} = te^t - e^t + \text{C}$$

$$\Rightarrow y\, e^{\tan x} = e^t(t-1) + \text{C}$$

Replacing t by $\tan x$ we get

$$ye^{\tan x} = e^{\tan x}(\tan x - 1) + \text{C} \qquad (1)$$

This is the general solution of the given equation

STEP 4 . Given that $y(1) = 0$, putting $x = 0$ and $y = 1$ in the general solution (1)we get

$$1.\ e^{\tan 0} = e^{\tan 0}(\tan 0 - 1) + \text{C}$$

$$\Rightarrow \text{C} = 1$$

STEP 5. Substitute the value of C in (1) we get the particular solution of the given differential equation as

$$ye^{\tan x} = e^{\tan x}(\tan x - 1) + 1$$

7.2.2 METHOD TO SOLVE A FIRST ORDER LINEAR DIFFERENTIAL EQUATION OF THE FORM II

$$\frac{dx}{dy} + \mathrm{P}(y) \times x = \mathrm{Q}(y)$$

STEP 1 . Write the given differential equation in the form $\frac{dx}{dy} + \mathrm{P}(y) \times x = \mathrm{Q}(y)$

STEP 2 . Obtain the values of $\mathrm{P}(x)$ and $\mathrm{Q}(x)$ from the equation of Step 1.

STEP 3 . Find the integrating Factor $(\mathrm{I.F.}) = e^{\int \mathrm{P}(y)dy}$

STEP 4 . Write the solution of the given differential equation as

$$x \times (\mathrm{I.F.}) = \int [\mathrm{Q}(y) \times (\mathrm{I.F.})]\, dy + \mathrm{C}$$

STEP 4 . If any initial condition is given like, at $x = 0, y = 2$, use it to find the value of C.

STEP 5. Substitute the value of C obtained in Step 4. to obtain the particular solution of the given differential equation.

EXAMPLE

Solve the initial value problem : $(1 + y^2)dx = (\tan^{-1}y - x)dy\,;\ \ y(0) = 0$

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SOLUTION

STEP 1 . The given differential equation can be written as

$$\frac{dx}{dy} + \left(\frac{1}{1+y^2}\right)x = \frac{\tan^{-1}y}{1+y^2}$$

STEP 2 . Here, $\mathrm{P}(y) = \frac{1}{1+y^2}$ and $\mathrm{Q}(y) = \frac{\tan^{-1}y}{1+y^2}$

STEP 3 . Integrating Factor $(\mathrm{I.F.}) = e^{\int \mathrm{P}(y)dy} = e^{\int \frac{1}{1+y^2}dy} = e^{\tan^{-1}y}$

STEP 4 . The solution of the given differential equation is given as

$$x \times (\mathrm{I.F.}) = \int [\,\mathrm{Q}(y) \times (\mathrm{I.F.})]\, dy + \mathrm{C}$$

$$\Rightarrow x \times \left(e^{\tan^{-1}y}\right) = \int \left[\left(\frac{\tan^{-1}y}{1+y^2}\right) \times \left(e^{\tan^{-1}y}\right)\right] dy + \mathrm{C}$$

Putting $\tan^{-1}y = t$ and $\frac{1}{1+y^2} = dt$ on RHS we get

$$xe^{\tan^{-1}y} = \int t\, e^t dt + \mathrm{C}$$

Integrating the integral by parts taking t as first function and e^t as second function we get

$$xe^{\tan^{-1}y} = t\int e^t\,dt - \int\left[\frac{d}{dt}(t)\int e^t dt\right]dt + C$$

$$\Rightarrow xe^{\tan^{-1}y} = te^t - \int e^t dt + C$$

$$\Rightarrow xe^{\tan^{-1}y} = te^t - e^t + C$$

$$\Rightarrow xe^{\tan^{-1}y} = e^t(t-1) + C$$

Replacing t by $\tan^{-1}y$ we get

$$xe^{\tan^{-1}y} = e^{\tan^{-1}y}(\tan^{-1}y - 1) + C \qquad (1)$$

This is the general solution of the given equation

STEP 5 . Given that $y(0) = 0$, putting $x = 0$ and $y = 0$ in the general solution (1)we get

$$0e^{\tan^{-1}0} = e^{\tan^{-1}0}(\tan^{-1}0 - 1) + C$$

$$\Rightarrow C = 1$$

STEP 6. Substitute the value of C in (1) we get the particular solution of the given differential equation as

$$xe^{\tan^{-1}y} = e^{\tan^{-1}y}(\tan^{-1}y - 1) + 1$$

$$\Rightarrow e^{\tan^{-1}y}(x - \tan^{-1}y + 1) = 1$$

VERY IMPORTANT : HOW TO RECOGNIZE THE GIVEN EQUATION IS OF WHICH TYPE

One of the major challenge in this chapter is to recognize that the equation given in the question is of which type. Students often get confused in recognizing the equation and due to this unable to solve it, in spite knowing all the three methods.

The following trick will help the students to quickly find the type of equation given:

Step 1. Just observe the given differential equation and carefully look for the following:

Case I If the differential equation contains only algebraic terms such that the degree of each term is same, then it is a Homogeneous Differential Equation and can be converted to any of the two forms.

Case II If the differential equation contains the terms like $e^{\frac{y}{x}}$, $\sin\left(\frac{y}{x}\right)$, $\log\left(\frac{y}{x}\right)$ etc., then it is a Homogeneous Differential Equation of the **FORM I :** $\frac{dy}{dx} = g\left(\frac{y}{x}\right)$

Case III If the differential equation contains the terms like $e^{\frac{x}{y}}$, $\sin\left(\frac{x}{y}\right)$, $\log\left(\frac{x}{y}\right)$ etc., then it is a Homogeneous Differential Equation of the **FORM II :** $\frac{dx}{dy} = h\left(\frac{x}{y}\right)$.

Step 2. Obtain the value of $\frac{dy}{dx}$ from the equation. If it is of the form $\frac{dy}{dx} = f(x) \times g(y)$, then the differential equation is a Variable Separable Equation.

Step 3. If the differential equation does not fall in the above two categories, then it must be a Linear differential Equation.

Case I If the differential equation contains only linear powers of x then it is of the

FORM I: $\frac{dy}{dx} + \mathrm{P}(x) \times y = \mathrm{Q}(x)$

Case II If the differential equation contains only linear powers of y then it is of the

FORM II: $\frac{dx}{dy} + \mathrm{P}(y) \times x = \mathrm{Q}(y)$