# FORMULAE & KEY POINTS
# CLASS 12 MATHEMATICS
# CHAPTER 06 /APPLICATIONS OF DERIVATIVES

**PRE-REQUISITE KNOWLWDGE FOR THE CHAPTER**

To solve the problems in this chapter, students must have the knowledge of the following topics of previous classes:

(i) Mensuration (Class X)
(ii) Functions (Class XI)
(iii) Coordinate Geometry (XI)
(iv) Polynomials (Class X)
(v) Solving a cubic equation by factor theorem and solving quadratic equation by factorization / Quadratic Formula (Classes IX & X)

## 1. RATE OF CHANGE OF QUANTITIES

**(i)** $\frac{dy}{dx}\left(or\ f'(x)\right)$ represents the rate of change of $y$ with respect to $x$

**(ii)** $\left.\frac{dy}{dx}\right]_{x=x_0}$ $\left(or\ f'(x_0)\right)$ represents the instantaneous rate of change of $y$ with respect to $x$ at $x = x_0$

**(iii)** If two variables $x$ and $y$ are varying with respect to another variable $t$, i.e.,

if $x = f(t)$ and $y = g(t)$, then by Chain Rule, $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$ provided $\frac{dx}{dt} \neq 0$

**(iv) Marginal Cost(MC)**

Let $C(x)$ be the Total Cost Function, then Marginal Cost is the instantaneous rate of change of total cost at any level of output.

Formula: $MC = \frac{dC}{dx}$

**(v) Marginal Revenue(MR)**

Let $R(x)$ be the Total Revenue Function, then Marginal Revenue is the the rate of change of total revenue with respect to the number of items sold at an instant

Formula: $MR = \frac{dR}{dx}$

**REMARK**

For a function $y = f(x)$

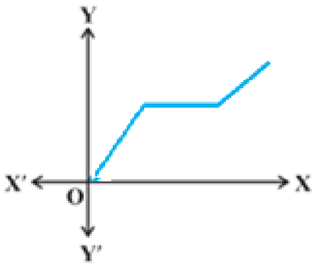
**(i)** if $y$ increasesas as $x$ increases, then $\frac{dy}{dx}$ is $+$ ve

**(ii)** if $y$ decreasesas as $x$ increases, then $\frac{dy}{dx}$ is $-$ ve

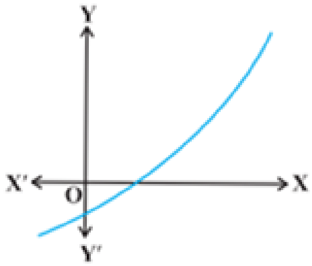
**(iii)** if $y$ does not change as $x$ increases, then , then $\frac{dy}{dx} = 0$

**(iv)** Rate of **change** of $y$ w.r.t. $x$ is $-$ 5 cm/ sec $\Leftrightarrow$ Rate of **decrease** of $y$ w.r.t. $x$ is 5 cm/sec.

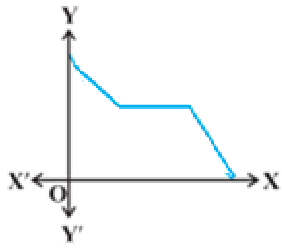
## 2.1 INCREASING AND DECREASING FUNCTIONS IN AN OPEN INTERVAL I



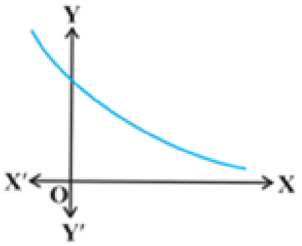
Increasing function
(i)



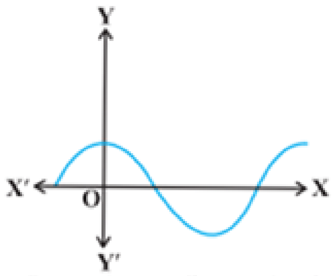
Strictly Increasing function
(ii)



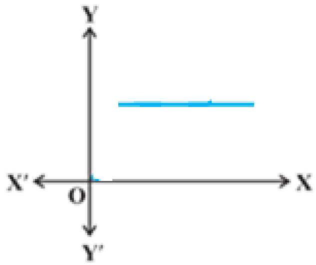
Decreasing function
(iii)



Strictly Decreasing function
(iv)



Neither Increasing nor Decreasing function
(v)



**Constant function**
**(vi)**

let I be an open interval contained in the domain of a real valued function f. Then f is said to be

**(i)** **Increasing** on I if $x_1 < x_2$ in I $\Rightarrow$ $f(x_1) \leq f(x_2)$ for all $x_1, x_2 \in$ I.

**(ii)** **Strictly Increasing** on I if $x_1 < x_2$ in I $\Rightarrow$ $f(x_1) < f(x_2)$ for all $x_1, x_2 \in$ I.

**(iii)** **Decreasing** on I if $x_1 < x_2$ in I $\Rightarrow$ $f(x_1) \geq f(x_2)$ for all $x_1, x_2 \in$ I.

**(iv)** **Strictly Decreasing** on I if $x_1 < x_2$ in I $\Rightarrow$ $f(x_1) > f(x_2)$ for all $x_1, x_2 \in$ I.

**(v)** **Neither Increasing Nor Decreasing** if sometimes $x_1 < x_2$ in I $\Rightarrow$ $f(x_1) \leq f(x_2)$ and For some other time $x_3 < x_4$ in I $\Rightarrow$ $f(x_3) \geq f(x_4)$ or $f$ is a constant function.

**(vi)** **A Constant Function** on I if $f(x_1) = f(x_2) = c$, a constant for all $x_1, x_2 \in$ I

## 2.2 INCREASING AND DECREASING FUNCTIONS AT A POINT

Let $x_0$ be a point in the domain of a real valued function $f$. Then f is said to be increasing, strictly increasing, decreasing or strictly decreasing at $x_0$ if there exists an open interval I containing $x_0$ such that f is increasing, strictly increasing, decreasing or strictly decreasing, respectively, in I.

Thus, a function $f$ is said to be increasing at $x_0$ if there exists an interval
$\text{I} = (x_0 - h, x_0 + h), h > 0$ such that for $x_1, x_2 \in$ I, $x_1 < x_2$ in I $\Rightarrow f(x_1) \leq f(x_2)$

Similarly, the other cases can be stated.

## 2.3 FIRST DERIVATIVE TEST FOR INCREASING AND DECREASING FUNCTION

If a function $f$ is continuous on the closed interval $[a, b]$ and differentiable on the open interval $(a, b)$. Then

**(i)** $f$ is increasing in $[a, b] \Leftrightarrow f'(x) \geq 0$ for each $x \in (a, b)$

**(ii)** $f$ is strictly increasing in $[a, b] \Leftrightarrow f'(x) > 0$ for each $x \in (a, b)$

**(iii)** $f$ is decreasing in $[a, b] \Leftrightarrow f'(x) \leq 0$ for each $x \in (a, b)$

**(iv)** $f$ is strictly decreasing in $[a, b] \Leftrightarrow f'(x) < 0$ for each $x \in (a, b)$

**(v)** $f$ is a constant function in $[a, b] \Leftrightarrow f'(x) = 0$ for each $x \in (a, b)$

**Example:**

The function $f(x) = \cos x$ is

(i) Strictly decreasing in $(0, \pi)$.
(ii) Decreasing in $[0, \pi]$ as $f$ is continuous in $[0, \pi]$
(iii) Strictly increasing in $(\pi, 2\pi)$
(iv) Increasing in $[\pi, 2\pi]$ as $f$ is continuous in $[\pi, 2\pi]$
(v) Neither increasing nor decreasing in $(0, 2\pi)$

### REMARKS

If a function $f$ is continuous on the closed interval $[a, b]$ and differentiable on the open interval $(a, b)$ and

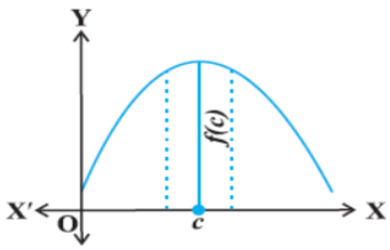
**(i)** $f'(x) > 0$ for all $x$ in $(a, b)$ except for a finite number of points where $f'(x) = 0$, then $f$ is strictly increasing in [a, b].

**(ii)** $f'(x) < 0$ for all $x$ in $(a, b)$ except for a finite number of points where $f'(x) = 0$, then $f$ is strictly decreasing in [a, b].

## 3. MAXIMUM VALUE, MINIMUM VALUE & EXTREME VALUE OF A FUNCTION IN AN INTERVAL

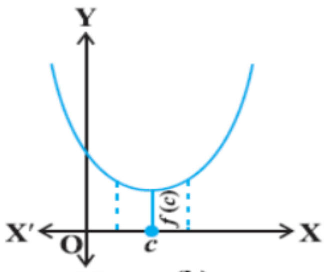
Let $f$ be a function defined on an interval I. Then

**3.1** $f$ is said to have a **maximum value** in I, if there exists a point $c$ in I such that $f(c) > f(x)$, for all $x \in$ I.
The number $f(c)$ is called the maximum value of $f$ in I and the point $c$ is called a **point of maximum value** of f in I.



(a)

**3.2** $f$ is said to have a **minimum value** in I, if there exists a point $c$ in I such that $f(c) < f(x)$, for all $x \in$ I.
The number $f(c)$ is called the minimum value of $f$ in I and the point $c$ is called a **point of minimum value** of $f$ in I.



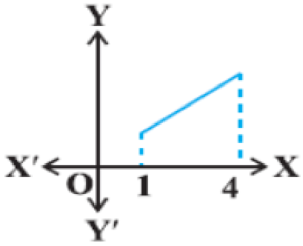
(b)

3.3 $f$ is said to have an **extreme value** in I if there exists a point $c$ in I such that $f(c)$ is either a maximum value or a minimum value of $f$ in I.
The number $f(c)$, in this case, is called an extreme value of $f$ in I and the point $c$ is called an **extreme point**.

## 4. MONOTONIC FUNCTION

**4.1** By a monotonic function $f$ in an interval I, we mean that $f$ is either increasing in I or decreasing in I.

**4.2** Every monotonic function assumes its maximum/minimum value at the end points of the domain of definition of the function.




**4.3** A more general result is
Every continuous function on a closed interval has a maximum and a minimum value.

## 5. NEIGHBOURHOOD(NBD) OF A REAL NUMBER/POINT ON A NUMBER LINE

**5.1** Let $c$ be a real number and let $h$ be a positive real number ( i.e. $h > 0$ ), then the Open Interval $(c - h, c + h)$ is called a neighbourhood of the point $c$, denoted as **nbd**$(\boldsymbol{c})$.

**5.2** The neighbourhood of a point on a real line is an interval, the neighbourhood of a point on a plane is a circle whereas the neighbourhood of a point space is a sphere.

## 6. TURNING POINTS OF A FUNCTION AND THEIR PROPERTIES

**6.1** A point on the graph of a function where the function changes its nature from decreasing to increasing or vice-versa is called a **turning point** of the function.
For example, in the graph of a function given below, the points A, B, C and D are the turning points of the function.

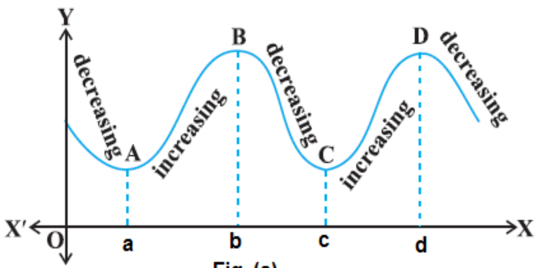


Fig. (a)

### 6.2 REMARKS

**(i)** At a turning point, say $x = c, f'(x) = 0$

**(ii)** In the neighbourhood of a turning point, the graph has either a little hill ( points B and D) or a little valley (points A and C).

**(iii)** A turning point is either a point of maximum value in its neighbourhood (points B and D) or a point of minimum value in its neighbourhood(points A and C)

**(iv)** In the above graph, the points A and C are called the **points of local minimum value (or relative minimum value)** and , the points B and D are called **the points of local maximum value (or relative maximum value).**

(v) The **local maximum value and local minimum value** of the function are referred to as **local maxima and local minima**, respectively, of the function.

**(vi)** For both the functions $\sin x$ and $\cos x$ the minimum value is $(-1)$and the maximum value is1 as the range of these functions is $[-1,1]$.

**(vii)** For both the functions $f(x) = a\sin x + b\cos x$ and $f(x) = a\sin x - b\cos x$, the minimum value is $(-\sqrt{a^2+b^2}\,)$ and the maximum value is $\sqrt{a^2+b^2}$.

**(viii)** Putting $a = b = 1$ in (vii)we get Thus, the minimum and the maximum values of both the functions $f(x) = a\sin x \pm b\cos x$ as $(-\sqrt{2})$ and $\sqrt{2}$ respectively.

**(ix)** For both the functions $\tan x$ and $\cot x$ the minimum value is $(-\infty)$and the maximum value is $(\infty)$ as the range of these functions is R.

**(x)** For both the functions $\sec x$ and $\operatorname{cosec} x$ the minimum value is $(-\infty)$and the maximum value is $(\infty)$ as the range of these functions is $(-\infty - 1] \cup [1,\infty)$.

**(xi)** For both the functions $f(x) = x^2$ and $f(x) = |x|$, the minimum value is 0 and the maximum value is $\infty$ as the range of these functions is $[0,\infty)$.

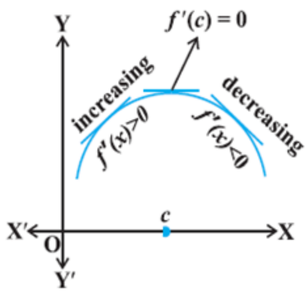
**(xii)** For the functions $f(x) = x^3$ the minimum value is $(-\infty)$and the maximum value is $(\infty)$ as the range of these functions is R

## 7. LOCAL MAXIMA AND LOCAL MINIMA

Let *f* be a real valued function and let $c$ be an interior point in the domain of *f*. Then

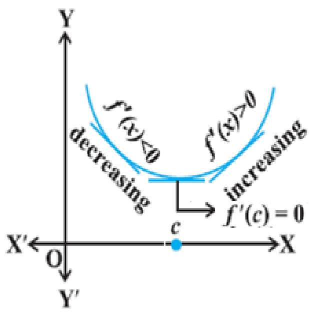
**7.1** $c$ is called a **point of local maxima** if there is an $h > 0$ such that $f(c) \geq f(x)$, for all $x$ in $(c-h, c+h)$

The value $f(c)$ is called the **local maximum value** of $f$.




**7.2** $c$ is called a **point of local minima** if there is an $h > 0$ such that $f(c) \leq f(x)$, for all $x$ in $(c-h, c+h)$

The value $f(c)$ is called the **local minimum value** of $f$.




### REMARKS

(i) Geometrically, if $x = c$ is a point of local maxima of $f$, then the function $f$ is increasing (i.e., $f'(x) > 0$)in the interval $(c-h, c)$ and decreasing ($i.e., f'(x) < 0$)in the interval $(c, c+h)$.

$\Rightarrow$ If $x = c$ is a point of local maxima then $f'(c)$ must be zero. But the converse is not true.

**(ii)** Geometrically, if $x = c$ is a point of local minima of $f$, then the function $f$ is decreasing (i.e., $f'(x) < 0$) in the interval $(c - h, c)$ and increasing ($i.e., f'(x) > 0$) in the interval $(c, c + h)$.

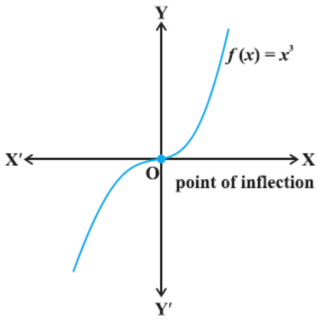
$\Rightarrow$ If $x = c$ is a point of local minima then $f'(c)$ must be zero. But the converse is not true.

## 7.3 THEOREM : CONDITION FOR A CRITICAL POINT

Let $f$ be a function defined on an open interval I. Suppose $c \in$ I be any point. If $f$ has a local maxima or a local minima at $x = c$, then,
either $f'(c) = 0$ **OR** $f$ is not differentiable at $c$

### REMARKS

**(i)** The converse of above theorem need not be true, that is, a point at which the derivative vanishes need not be a point of local maxima or local minima.
For example, if $f(x) = x^3$, then $f'(x) = 3x^2$ and so $f(0) = 0$. But 0 is neither a point of local maxima nor a point of local minima.



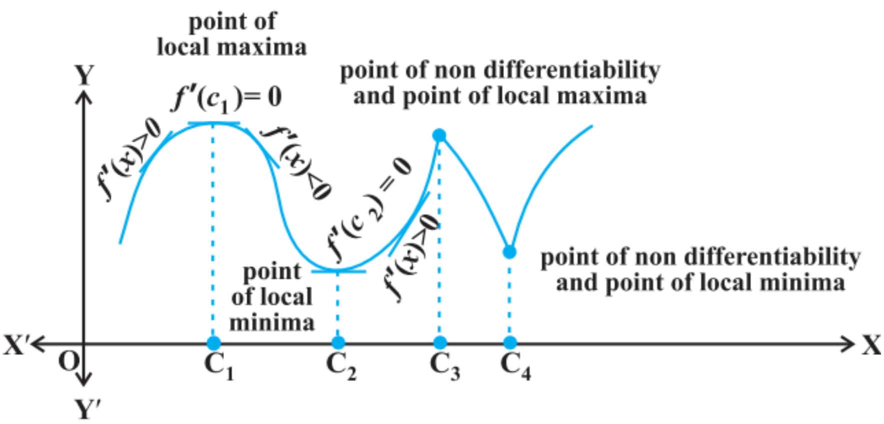

**(ii)** **CRITICAL POINTS OF A FUNCTION**

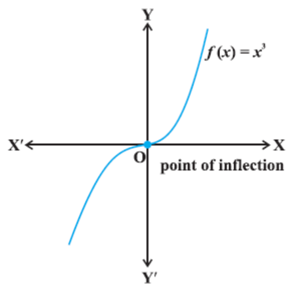
A point $c$ in the domain of a function $f$ at which either $f'(c) = 0$ or $f$ is not differentiable is called a **critical point** of $f$. Note that if $f$ is
continuous at $c$ and $f'(c) = 0$, then there exists an $h > 0$ such that $f$ is differentiable in the interval $(c - h, c + h)$.

**(ii)** **POINT OF INFLECTION OF A FUNCTION**

Let $f$ be a function defined on an open interval I. Let $f$ be continuous at a critical point $c$ in I. Then, if $f'(x)$ does not change sign as $x$ increases through $c$, then $c$ is neither a point of local maxima nor a point of local minima. Such a point is called **point of inflection**.
Thus, in the above example, $x = 0$ is a point of inflection of the function $f$.

## 8. FIRST DERIVATIVE TEST FOR FINDING LOCAL MAXIMA AND LOCAL MINIMA OF A FUNCTION



Let $f$ be a function defined on an open interval I. Let $f$ be continuous at a critical point $c$ in I. Then

**(i)** If $f'(x)$ changes sign from positive to negative as $x$ increases through $c$, i.e., if $f'(x) > 0$ at every point sufficiently close to and to the left of $c$, and $f'(x) < 0$ at every point sufficiently close to and to the right of $c$, then $c$ is a **point of local maxima**.

**(ii)** If $f'(x)$ changes sign from negative to positive as $x$ increases through $c$, i.e., if $f'(x) < 0$ at every point sufficiently close to and to the left of $c$, and $f'(x) > 0$ at every point sufficiently close to and to the right of $c$, then $c$ is a **point of local minima.**

**(iii)** If $f'(x)$ does not change sign as $x$ increases through $c$, then $c$ is neither a point of local maxima nor a point of local minima. In fact, such a point is called **point of inflection.**

### REMARKS

**(i)** If $c$ is a point of local maxima of $f$ , then $f(c)$ is a local maximum value of $f$.

**(ii)** If $c$ is a point of local minima of $f$ , then $f(c)$ is a local minimum value of $f$

## 9. SECOND DERIVATIVE TEST FOR FINDING LOCAL MAXIMA AND LOCAL MINIMA OF A FUNCTION

Let $f$ be a function defined on an interval I and $c \in$ I. Let $f$ be twice differentiable at $c$. Then

**(i)** $x = c$ is a point of local maxima if $f'(c) = 0$ and $f''(c) < 0$

In this case, $f(c)$ is local maximum value of $f$.

**(ii)** $x = c$ is a point of local minima if $f'(c) = 0$ and $f''(c) > 0$

In this case, $f(c)$ is local minimum value of $f$ .

**(iii)** The test fails if $f'(c) = 0$ and $f''(c) = 0$.

In this case, we go back to the first derivative test and find whether $c$ is a point of local maxima, local minima or a point of inflexion

## 10. ABSOLUTE MAXIMUM VALUE (GLOBAL MAXIMUM OR GREATEST VALUE) AND ABSOLUTE MINIMUM VALUE (GLOBAL MINIMUM OR LEAST VALUE) OF A FUNCTION IN A CLOSED INTERVAL

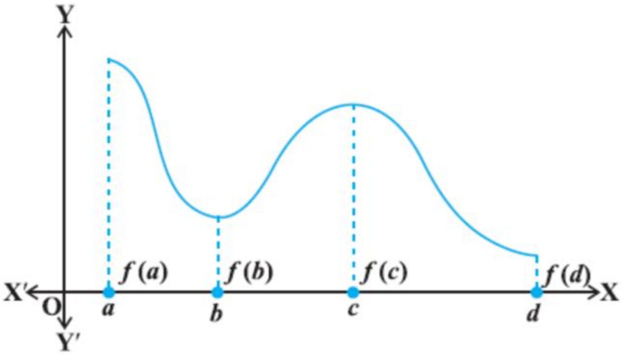
### 10.1 MAXIMUM OR ABSOLUTE MAXIMUM OF A FUNCTION

A function $f$ defined over $[a, b]$ is said to have maximum (or absolute maximum) at $x = c, c \in [a, b], if\ f(x) \leq f(c)$ for all $x \in [a, b]$.

### 10.2 MINIMUM OR ABSOLUTE MINIMUM OF A FUNCTION

A function f (x) defined over [a, b] is said to have a minimum [or absolute minimum] at $x = d$, if $f(x) \geq f(d)$ for all $x \in [a, b]$.

**10.3**




**(i)** Consider the graph given in the Fig of a continuous function defined on a closed interval $[a, b]$. Observe that the function $f$ has a local minima at $x = b$ and local minimum value is $f(b)$. The function also has a local maxima at $x = c$ and local maximum value is $f(c)$.

**(ii)** Also from the graph, it is evident that $f$ has absolute maximum value $f(a)$ and absolute minimum value $f(d)$. Further note that the absolute maximum (minimum) value of $f$ is different from local maximum (minimum) value of $f$.

**(iii)** Let $f$ be a continuous function on an interval I = $[a, b]$. Then f has the absolute maximum value and $f$ attains it at least once in I. Also, $f$ has the absolute minimum value and attains it at least once in I.

**(iv)** Let $f$ be a differentiable function on a closed interval I and let $c$ be any interior point of I. Then

**(a)** $f'(c) = 0$ if $f$ attains its absolute maximum value at $c$.

**(b)** $f'(c) = 0$ if $f$ attains its absolute minimum value at $c$.

## 10.4 WORKING RULE FOR FINDING ABSOLUTE MAXIMA AND OR ABSOLUTE MINIMA

**Step 1:** Find all the **critical points** of $f$ in the given interval and the **end points** of the given closed interval.

**Step 2:** At all these points obtained in Step 1, calculate the values of $f$.

**Step 3:** Identify the maximum and minimum values of $f$ out of the values calculated in Step 2.

The maximum value will be the absolute maximum value of f and the minimum value will be the absolute minimum value of $f$