FORMULAE & KEY POINTS

CLASS 11 MATHEMATICS
CHAPTER 11: LIMITS & DERIVATIVES
CLASS 12 MATHEMATICS
CHAPTER 05 : CONTINUITY AND DIFFERENTIABILITY
PART I : LIMITS

1. MEANING OF $x \to a$

1.1 $x \to a$ is read as "x approaches to a" or " x tends to a" and it means x comes arbitrarily nearer to a but is not equal to a.

1.2 $x \to a \Leftrightarrow (x-a) \to 0$

1.3 $x \to 0 \Leftrightarrow ax \to 0$

2. LIMIT OF A FUNCTION $y = f(x)$ AT $x = a$ (OR SIMPLY LIMIT OF FUNCTION f AT a)

2.1 Left Hand Limit (LHL) of f at point a

The expected value of the function $y = f(x)$ as x approaches toward a from the left hand side of a is called the Left Hand Limit (LHL) of f at $x = a$ and is denote as $\lim\limits_{x \to a^-} f(x)$.

2.2 Right Hand Limit RHLof f at point a

The expected value of the function $y = f(x)$ as x approaches toward a from the right hand side of a is called the Reft Hand Limit (RHL) of f at $x = a$ and is denote as $\lim\limits_{x \to a^+} f(x)$.

2.3 Limit of f at point $x = a$

If the RHL and LHL of a function coincide at a point $x = a$, we call that common value as the limit of f at $x = a$ and denote it by $\lim\limits_{x \to a} f(x)$

3. ALGEBRA OF LIMIT

Let f and g be two functions such that both $\lim\limits_{x \to a} f(x)$ and $\lim\limits_{x \to a} g(x)$ exist. Then

3.1 Limit of sum of two functions is sum of the limits of the functions, i.e.,

$$\lim_{x \to a} [f(x) + g(x)] = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$$

3.2 Limit of difference of two functions is difference of the limits of the functions, i.e.,

$$\lim_{x \to a} [f(x) - g(x)] = \lim_{x \to a} f(x) - \lim_{x \to a} g(x)$$

3.3 Limit of product of two functions is product of the limits of the functions, i.e.,

$\lim_{x\to a} [f(x).g(x)] = \lim_{x\to a} f(x).\lim_{x\to a} g(x)$

In particular, for a real number λ,

$\lim_{x\to a} [\lambda.f(x)] = \lambda.\lim_{x\to a} f(x)$

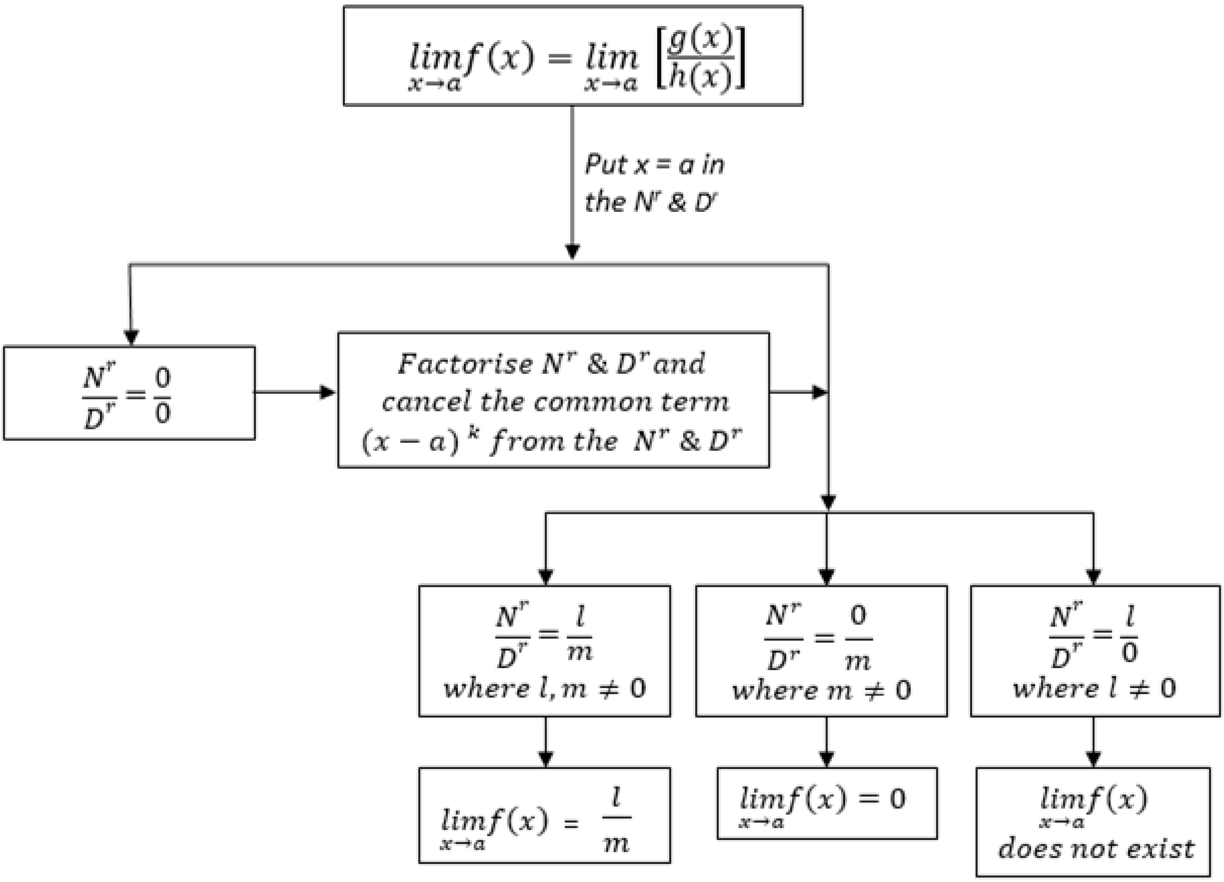
3.4 Limit of quotient of two functions is quotient of the limits of the functions (whenever the denominator is not zero), i.e., $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$

4. Limit of a Polynomial Function $p(x)$

$\lim_{x\to a} p(x) = p(a)$

5. Algorithm to Find Limit of a Rational Function

Let Let $y = f(x) = \frac{g(x)}{h(x)}$, be a rational function (i.e. g(x) and h(x) are polynomials and h(x) $\neq$ 0).



6. For any positive integer n, $\lim_{x\to 0} \frac{x^n - a^n}{x-a} = na^{n-1}$ where a

REMARK

The above formula is also true for $a > 0$ and n is a rational number

7. TRIGONOMETRIC LIMITS

7.1 $\lim\limits_{x \to 0} \frac{\sin x}{x} = 1$

REMARKS

(i) $\lim\limits_{x \to 0} \frac{\sin ax}{x} = \lim\limits_{x \to 0} \left(\frac{\sin ax}{ax}\right) \times a = 1.a = a$

(ii) $\lim\limits_{x \to 0} \frac{\sin^m x}{x^m} = \lim\limits_{x \to 0} \left(\frac{\sin x}{x}\right)^m = 1^m = 1$

7.2 $\lim\limits_{x \to 0} \frac{\tan x}{x} = 1$

7.3 $\lim\limits_{x \to 0} \frac{1 - \cos x}{x} = 0$

8. EXPONENTIAL LIMIT

(i) $\lim\limits_{x \to 0} \frac{e^x - 1}{x} = 1$

(ii) $\lim\limits_{x \to 0} \frac{e^{ax} - 1}{ax} = 1$

8. LOGARITHMIC LIMIT

8.1 $\lim\limits_{x \to 0} \frac{\log(1 + x)}{x} = 1$

8.2 $\lim\limits_{x \to 0} \frac{\log(1 + ax)}{ax} = 1$

PART II : CONTINUITY

1. CONTINUOUS FUNCTION

Suppose f is a real function on a subset of the real numbers and let c be a point in the domain of f. Then f is continuous at $x = c$ if

$$\lim_{x \to c} f(x) = f(c)$$

$\boldsymbol{i.e.}$ [LHL at c] = [RHL at c] = [Value of Function f at c]

$\boldsymbol{i.e.}$ $\lim_{x \to c^-} f(x) = \lim_{x \to c^+} f(x) = f(c)$

REMARK

A continuous function is that whose graph can be drawn without lifting the pen from the paper.

2. DISCONTINUOUS FUNCTION

A function $y = f(x)$is said to be discontinuous at a point $x = c$ if the condition $\lim_{x \to c^-} f(x) = \lim_{x \to c^+} f(x) = f(c)$ is not satisfied.

Thus, a function $y = f(x)$is discontinuous at $x = c$ if any of the following conditions is satisfied:

(i) LHL does not exist

Example

For the function $f(x) = \begin{cases} \frac{1}{x} \text{ if } x < 0 \\ 1 \text{ if } x \geq 0 \end{cases}$, at $x = 0$ LHL does not exists wheresas RHL $= 1$

OR

(ii) RHL does not exist

Example

For the function $f(x) = \begin{cases} \frac{1}{x} \text{ if } x \geq 0 \\ 1 \text{ if } x < 0 \end{cases}$, at $x = 0$ RHL does not exists wheresas LHL $= 1$

OR

(iii) RHL and RHL both exist but LHL $\neq$ RHL

Example

For the Signum Function $f(x) = \begin{cases} -1 \text{ if } x < 0 \\ 1 \text{ if } x > 0 \\ 0 \;\; if \;\; x = 0 \end{cases}$

at $x = 0$, LHL $= -1$ whereas RHL $= 1$ $\therefore$ LHL and RHL both exist but LHL $\neq$ RHL

OR

(iv) $\lim_{x \to c} f(x)$ exists but $\lim_{x \to c} f(x) \neq f(c)$

Example

For the function, $f(x)=\begin{cases}\dfrac{x^2-4}{x-2} \text{ if } x\neq 0\\ 3 \text{ if } x=2\end{cases}$, $\lim\limits_{x\to 2} f(x)=4$ but $\lim\limits_{x\to 2} f(x)\neq f(2)$

REMARK

A real function f is said to be continuous if it is continuous at every point in the domain of f.

3. THE FOLLOWING FUNCTIONS ARE CONTINUOUS (i.e. CONTINUOUS IN THEIR RESPECTIVE DOMAINS)

[**NOTE**: *To know about the Definition, Domain, Range and Graph of these functions, refer Formulae Sheet of Class XI – Chapter Relations and Functions.*]

(i) Polynomial Functions (.i.e. Constant Function, Linear Function, Quadratic Function etc)
$p(x)=a_0+a_1x+a_2x^2+a_3x^3+a_4x^4+\ .\ .\ .\ +a_nx^n$

(ii) Rational Functions : $f(x)=\dfrac{p(x)}{q(x)}$ where $p(x)$and $q(x)$ are polynomials and $q(x)\neq 0$.

(iii) Modulus Function : $f(x)=|x|$

(iv) Trigonometric Functions : $\sin x, \cos x, \tan x, \cot x, \sec x, cosec\ x$

(v) Inverse Trigonometric Functions : $\sin^{-1}x, \cos^{-1}x, \tan^{-1}x, \cot^{-1}x, \sec^{-1}x, cosec^{-1}x$

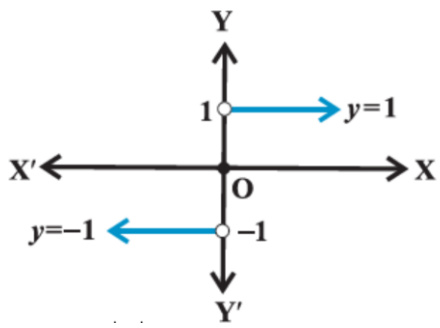
(vi) Exponential Functions : $f(x)=e^x$ and a^x

(vii) Logarithmic Functions : $f(x)=log\ x$

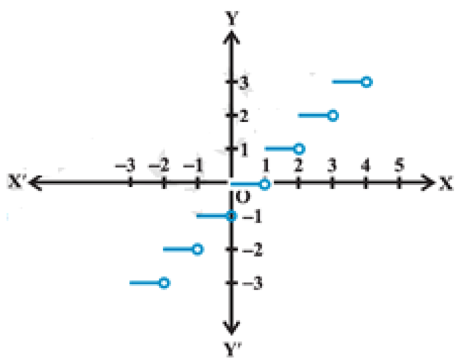
4. THE FOLLOWING FUNCTIONS ARE DISCONTINUOUS

(i) Signum Function: $f(x)=\begin{cases}-1 \text{ if } x<0\\ 0 \text{ if } x=0\\ 1 \text{ if } x>0\end{cases}$

The Signum Function is discontinuous at $x=0$.
as at x = 0, LHL = −1, RHL = 1 and $f(0)=0$



(ii) Greatest Integer Function: $f(x)=[x]$, where $[x]$ is the greatest integer greater than of equal to x.
The Greatest Integer Function is discontinuous at The integral points $0, \pm1, \pm2, \pm3, .\ .\ .$ as for an integer, say $x=n$,
LHL = (n − 1) whereas RHL = $f(n)=n$



6. ALGEBRA OF CONTINUOUS FUNCTIONS

Suppose f and g be two real functions continuous at a real number c. Then

(i) $f + g$ is continuous at $x = c$.

(ii) $f - g$ is continuous at $x = c$.

(iii) $f . g$ is continuous at $x = c$

(iv) $\left(\frac{f}{g}\right)$ is continuous at $x = c$, provided $g(c) \neq 0$.

7. CONTINUITY OF COMPOSITION OF TWO FUNCTIONS

Suppose f and g are real valued functions such that $(fog)$is defined at c. If g is continuous at c and if f is continuous at $g(c)$, then (fog) is continuous at c.

PART III : DERIVATIVES

1. DERIVATIVE OF A FUNCTION $y = f(x)$ at $x = a$

Suppose f is a real valued function and $\boldsymbol{a}$ is a point in its domain of definition.
The derivative of f at $x = a$ is defined as $\lim_{h\to 0}\dfrac{f(a+h)-f(a)}{h}$ provided this limit exists.

Derivative of $y = f(x)$ at $x = a$ is denoted by $f'(a)$ OR $\left[\dfrac{df}{dx}\right]_{x=a}$ OR $\left[\dfrac{dy}{dx}\right]_{x=a}$ OR $y_1(a)$

2. DERIVATIVE OF A FUNCTION $y = f(x)$ at x

Suppose f is a real valued function and $\boldsymbol{x}$ is a point in its domain of definition.
The derivative of f at x is defined as $\lim_{h\to 0}\dfrac{f(x+h)-f(x)}{h}$ provided this limit exists.

Derivative of $y = f(x)$ at x is denoted by $f'(x)$ OR $\dfrac{df}{dx}$ OR $\dfrac{dy}{dx}$ OR $y_1(x)$ OR Dy OR y′

REMARKS

(i) The derivative of a function obtained by using the above formula is called
(**a**) Differention by First Principal **OR**
(**b**) Differention by Using Definition **OR**
(**c**) Differentiation ab − initio **OR**
(**d**) Differention by Delta Method

(ii) The Derivative of a Function $y = f(x)$ is also given by the following expressions

$$\frac{dy}{dx} = \lim_{x\to 0}\frac{\Delta y}{\Delta x} = \lim_{h\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x} = \lim_{h\to 0}\frac{f(x+\Delta\delta)-f(x)}{\Delta\delta}$$

3. DIFFERENTIABILITY OF A FUNCTION IN AN INTERVAL

(i) A function is said to be differentiable in an interval [a, b] if it is differentiable at every point of [a, b].

(ii) A function is said to be differentiable in an interval (a, b) if it is differentiable at every point of (a, b)

4. THEOREM

If a function f is differentiable at a point c, then it is also continuous at that point.
But the converse of the theorem is not true.
For Example, the modulus function *f(x)=|x|* is continuous but not differentiable at *x* = 0.

Corollary: Every differentiable function is continuous. (But a continuous function may or may not be differentiable).

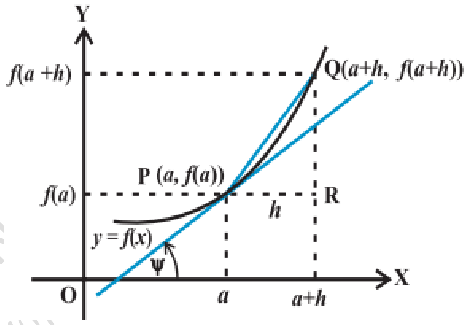
5. PHYSICAL INTERPRETATION/ MEANING OF DERIVATIVE OF A FUNCTION

Physically, the derivative of a function $y = f(x)$ at a point $x = a$ represents the Instantaneous Rate of Change of $y = f(x)$ at the instant $x = a$

Example: Velocity $=$ Rate of change of displacement w.r.t. time $= \dfrac{ds}{dt}$

6. GEOMETRICAL INTERPRETATION/ MEANING OF DERIVATIVE OF A FUNCTION

Geometrically, the derivative of a function $y = f(x)$ at a point $x = a$ represents the slope of the tangent to the curve $y = f(x)$ at the point $(a, f(a))$.



7. ALGEBRA OF DERIVATIVE OF FUNCTIONS

Let f and g be two functions such that their derivatives are defined in a common domain.

7.1 Addition Rule of Differentiation

Derivative of Sum $=$ Sum of the Derivatives

$$\frac{d}{dx}[f(x) + g(x)] = \frac{d}{dx}f(x) + \frac{d}{dx}g(x)$$

7.2 Subtraction Rule of Differentiation

Derivative of Difference $=$ Difference of the Derivatives

$$\frac{d}{dx}[f(x) + g(x)] = \frac{d}{dx}(f(x)) + \frac{d}{dx}(g(x))$$

7.3 Product Rule OR Leibnitz Rule of Differentiation

7.3.1 (Derivative of the Product of Two Functions)

$= \left[(2^{\text{nd}} \text{ Fun.}) \times (\text{Derivative of } 1^{\text{st}} \text{ Fun.})\right] + \left[(1^{\text{st}} \text{ Fun.}) \times (\text{Derivatiove of } 2^{\text{nd}} \text{ Fun.})\right]$

Symbolically,

$$\frac{d}{dx}[f(x).g(x)] = g(x) \times \frac{d}{dx}f(x) + f(x) \times \frac{d}{dx}g(x) \quad \textbf{OR} \quad (uv)' = vu' + uv'$$

7.3.2 Example

Mistake: $\dfrac{d}{dx}(x^3.\tan x) \neq 3x^2.\sec^2 x$

Correction: $\dfrac{d}{dx}(x^3.\tan x) = \tan x.(3x^2) + x^3 \sec^2 x$

(b) Derivative of the Product of Three Functions

$$(uvw)' = uvw' + uv'w + u'vw \quad \textbf{OR} \quad \frac{(uvw)'}{uvw} = \frac{u'}{u} + \frac{v'}{v} + \frac{w'}{w}$$

(c) Derivative of the Product of a function $y = f(x)$ with a scaler λ

$$\frac{d}{dx}[\lambda . f(x)] = \lambda . \frac{d}{dx} f(x)$$

Example

$$\frac{d}{dx}(25x^n) = 25\frac{d}{dx}(x^n)$$

7.4 Quotient or Division Rule of Differentiation:

(Derivative of the Quotient of Two Functions)

$$= \frac{[(\mathrm{D^r}) \times (\text{Derivative of } \mathrm{N^r})] - [(\mathrm{N^r}) \times (\text{Derivatiove of } \mathrm{D^r})]]}{[\mathrm{D^r}]^2}$$

Symbolically,

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x) \times \frac{d}{dx} f(x) - f(x) \times \frac{d}{dx} g(x)}{[g(x)]^2} \quad \textbf{OR} \quad \left(\frac{u}{v}\right)' = \frac{vu' - uv'}{v^2}$$

Example:

Mistake: $\frac{d}{dx}\left(\frac{x^3}{\tan x}\right) \neq \frac{3x^2}{\sec^2 x}$ **Correction:** $\frac{d}{dx}\left(\frac{x^3}{\tan x}\right) = \frac{\tan x . (3x^2) - x^3 \sec^2 x}{\tan^2 x}$

8. DERIVATIVES OF SOME STANDARD FUNCTIONS

	FUNCTIONS $[y = f(x)]$	DERIVATIVES $\left[\frac{dy}{dx} = f'(x)\right]$
8.1	**DERIVATIVES OF ALGEBRAIC FUNCTIONS**	
(i)	c, a constant	$\frac{d}{dx}(c) = 0$ **Examples** **Mistakes:** $\frac{d}{dx}[(100)^{99}] = 99.(100)^{98}$; $\frac{d}{dx}(a^n) = na^{n-1}$ **Corrections:** $\frac{d}{dx}((100)^{99}) = 0$; $\frac{d}{dx}(a^n) = 0$
(ii)	x	$\frac{d}{dx}(x) = 1$
(iii)	x^n	$\frac{d}{dx}(x^n) = nx^{n-1}$ **Example:** $\frac{d}{dx}(x^6) = 6x^5$

(iv)	$(ax+b)^n$	$\frac{d}{dx}(ax+b)^n = n(ax+b)^{n-1}.a$ **[By Chain Rule]** **Example:** $\frac{d}{dx}(3-4x)^5 = 5(3-4x)^5.(-4)$
(v)	$\frac{1}{x}$	$\frac{d}{dx}\left(\frac{1}{x}\right) = -\frac{1}{x^2}$
(vi)	$\sqrt{x}$	$\frac{d}{dx}(\sqrt{x}) = \frac{1}{2\sqrt{x}}$ $\quad (as\ \sqrt{x} = x^{1/2})$
(vii)	$\frac{1}{x^n}$ OR x^{-n}	$\frac{d}{dx}\left(\frac{1}{x^n}\right) = -\frac{n}{x^{n+1}} = -nx^{-(n+1)}$ **Example :** $\frac{d}{dx}\left(\frac{1}{x^4}\right) = \frac{d}{dx}(x^{-4}) = -\frac{4}{x^5} = -4x^{-5}$
8.2	**DERIVATIVES OF MODULUS FUNCTIONS**	
	$\lvert x\rvert,\ x \neq 0$	$\frac{d}{dx}(\lvert x\rvert) = \frac{x}{\lvert x\rvert},\ x \neq 0$
	REMARKS (**i**) The derivative of $f(x) = \lvert x\rvert$does not exist at $x = 0$. (**ii**) In general, the derivative of $f(x) = \lvert x+2\rvert + \lvert 2x-3\rvert$does nor exist at $x = -2$ and $x = 3/2$	
8.3	**DERIVATIVES OF TRIGONOMETRIC FUNCTIONS**	
(i)	$\sin x$	$\frac{d}{dx}(sinx) = \cos x$
(ii)	$\cos x$	$\frac{d}{dx}(\cos x) = -\sin x$
(iii)	$\tan x$	$\frac{d}{dx}(\tan x) = \sec^2 x$
(iv)	$\cot x$	$\frac{d}{dx}(\cot x) = -\mathrm{cosec}^2 x$
(v)	$\sec x$	$\frac{d}{dx}(\sec x) = \sec x \,.\tan x$
(vi)	$\mathrm{cosec}\, x$	$\frac{d}{dx}(\mathrm{cosec}\, x) = -\mathrm{cosec}\, x \,.\cot x$
	REMARK Derivatives of the trigonometric functions starting with the letter '**c**' have ' − ' signs..	
8.4	**DERIVATIVES OF EXPONEMTIAL AND LOGARITHMIC FUNCTIONS**	
(i)	a^x	$\frac{d}{dx}(a^x) = a^x \log a$
(ii)	e^x	$\frac{d}{dx}(e^x) = e^x$

(iii)	$\log x$ **OR** $\log\lvert x\rvert$	$\frac{d}{dx}(\log x) = \frac{d}{dx}(\log\lvert x\rvert) = \frac{1}{x}$
(iv)	$\log_a x$	$\frac{d}{dx}(\log_a x) = \frac{d}{dx}(\log_a\lvert x\rvert) = \frac{1}{x}$ $\left[\text{since} \log_a x = \frac{\log_e x}{\log_e a}\right]$
8.5	**DERIVATIVES OF INVERSE TRIGONOMETRIC FUNCTIONS**	
(i)	$\sin^{-1} x$	$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$
(ii)	$\cos^{-1} x$	$\frac{d}{dx}(\cos^{-1} x) = \frac{-1}{\sqrt{1-x^2}}$
(iii)	$\tan^{-1} x$	$\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$
(iv)	$\cot^{-1} x$	$\frac{d}{dx}(\cot^{-1} x) = \frac{-1}{1+x^2}$
(v)	$\sec^{-1} x$	$\frac{d}{dx}(\sec^{-1} x) = \frac{1}{x\sqrt{x^2-1}}$
(vi)	$\text{cosec}^{-1} x$	$\frac{d}{dx}(\text{cosec}^{-1} x) = \frac{-1}{x\sqrt{x^2-1}}$

REMARK

Derivatives of the inverse trigonometric functions starting with the letter '**c**' have '**–**' signs.

9 SOME SPECIAL METHODS OF DIFFERENTIATION

9.1 DERIVATIVES OF COMPOSITE FUNCTIONS BY CHAIN RULE OF DIFFERENTIATION

9.1.1 Chain Rule for the composition of two functions

Let f be a real valued function which is a composition of two functions u and v; i.e., $f = v \, o \, u$.

Suppose $t = u(x)$ so that $f = v(t)$ and if both $\frac{dt}{dx}$ and $\frac{dv}{dt}$ exist, then

$$\frac{df}{dx} = \left(\frac{dv}{dt}\right).\left(\frac{dt}{dx}\right)$$

EXAMPLE

Let $f(x) = \log(\tan x)$. Then f is the composition of two functions $\log x$ and $\tan x$.

Let, $t = u(x) = \tan x$ and $f = v(t) = \log t$, then, $\frac{df}{dx} = \left(\frac{dv}{dt}\right).\left(\frac{dt}{dx}\right) = \frac{1}{t}.\sec^2 x$

$$= \frac{1}{tanx}.\sec^2 x$$

9.1.2 Chain Rule for the composition of Three functions

Let f be a real valued function which is a composite of three functions u, v and w; i.e., $f = w\, o\, v\, o\, u$.

Suppose $t = u(x), z = v(t)$ and $f = w(z)$ and if all $\frac{dt}{dx}, \frac{dv}{dt}$ and $\frac{dw}{dz}$ exist, then

$$\frac{df}{dx} = \left(\frac{dw}{dz}\right)\left(\frac{dz}{dt}\right).\left(\frac{dt}{dx}\right)$$

EXAMPLE

Let $f(x) = \log(\tan(x^2 - 3))$.

Then f is the composition of two functions $\log x, \tan x$ and $x^2 - 3$.

Let, $t = u(x) = x^2 - 3, z = v(t) = \tan t$ and $f = w(z) = \log z$, then,

$$\frac{df}{dx} = \left(\frac{dw}{dz}\right)\left(\frac{dz}{dt}\right).\left(\frac{dt}{dx}\right) = \frac{1}{z}.\sec^2 t\,.2x = \frac{1}{\tan\,(x^2 - 3)}.\sec^2(x^2 - 3).2x$$

VERY IMPORTANT REMARK

The chain rule of differentition as explained and applied above may look to be complicated.

To apply it conveniently, it is suggested to apply the chain rule directly without making the substitutions in the form of u, v, w etc. In this, we start with the outermost function and move towards the innermost function, step by step, writing derivatives of the component functions of the composite function. Once we reach the innermost function we stop the process.

For Example

$$\frac{d}{dx}(\tan(\log(x^3 + 2x - 1) =$$

$$\sec^2[\log(x^3 + 2x - 1)] \quad \times \quad \frac{1}{(x^3 + 2x - 1)} \quad \times \quad (3x^2 + 2)$$

(derivative of $\tan t$ w.r.t. t taking $\log(x^3 + 2x - 1) = t$) (derivative of $\log z$ w.r.t. z taking $(x^3 + 2x - 1) = z$) (derivative of $x^3 + 2x - 1$ w.r.t. x)

PRECAUTIONS

1. The readers should make themselves very clear with the structure of the function as when to apply chain rule and when to apply product rule or when to apply both as given in the following examples:

(i) For $f(x) = \sin(\tan x)$ apply chain rule but for $f(x) = \sin x.\tan x$, apply product rule

(ii) For $f(x) = \log(\tan x)$ apply chain rule but for $f(x) = \log x.\tan x$, apply product rule

2. **Example 1:** In the function $f(x) = \log(\cos x^2)$, the outermost function is **log** function, next is **cos** function and the last is $\boldsymbol{x^2}$ function

Example 2: In the function $f(x) = \cos x^2$, the outermost function is **cos** function and the next is $\boldsymbol{x^2}$ function.

But, in the function $f(x) = \cos^2 x$, the outermost function is '**square function**' and the next is **cos** function as $\cos^2 x = (\cos x)^2$.

9.2. EXPLICIT AND IMPLICIT FUNCTIONS

(i) Explicit Function

When the relationship between x and y is expressed in a way that it is easy to solve for y and write $y = f(x)$, we say that y is an explicit function of x.

Example: $x - y - \pi = 0$

(ii) Implicit Function

When the relationship between x and y is expressed in a way that it is not easy to solve for y and write $y = f(x)$, we say that y is an implicit function of x.

Example: $x + \sin xy - y = 0$

(iii) METHOD TO FIND DERIVATIVE OF IMPLICIT FUNCTIONS

Step I : Differentiate the whole equation w.r.t. x

Step II : On LHS collect the terms containing $\frac{dy}{dx}$ and on RHS collects the terms not containing $\frac{dy}{dx}$

Step III : Take $\frac{dy}{dx}$ common from the terms on LHS

Step IV : Obtain the value of $\frac{dy}{dx}$ in the form $\frac{dy}{dx} = f(x, y)$

Example: Find $\frac{dy}{dx}$ if $y = xsiny$

Solution :

Step I : Differentiate the whole equation w.r.t. x we get $\frac{dy}{dx} = siny.1 + x\, siny.\frac{dy}{dx}$

Step II : $\frac{dy}{dx} - x\, siny.\frac{dy}{dx} = siny$

Step III : $\frac{dy}{dx}(1 - x\, siny) = siny$

Step IV : $\frac{dy}{dx} = \frac{siny}{(1 - x\, siny)}$

9.3. LOGARITHMIC DIFFERENTIATION

The Method of logarithmic differentiation is applied to the following types of functions:

(i) When $f(x)$ is the product and quotient of a number of functions.

Example: $f(x) = \sqrt{\frac{(x-1)(x-2)}{(x-30(x-4)(x-5)}}$

(ii) When the function contains terms of exponential form $[u(x)]^{v(x)}$ where base and expone both non − constant functions.

Example: **(i)** $f(x) = x^x$ **(ii)** $(\cos x)^y + (sinx)^y = a^b$

PRECAUTION : Students should be very careful about the following very common mistake done while using the method of logarithmic differentiation:

Mistake : $x^y + y^x = x^x \Rightarrow logx^y + logy^x = logx^x \Rightarrow ylogx + xlogy = xlogx$

Correction : $x^y + y^x = x^x \Rightarrow \frac{d}{dx}(x^y) + \frac{d}{dx}(y^x) = \frac{d}{dx}(x^x)$

9.4. PARAMETRIC FORM OF AN EQUATION

(i) The form of a cartesian equation in which the variables x and y are expressed in terms of a third variable, say t or θ, is called the parametric form of the equation and the third variable is called the PARAMETER.

Examples: (i) The parametric form of the parabola $y^2 = 4ax$ is $x = at^2; y = 2at$

(ii) The parametric form of the circle $x^2 + y^2 = r^2$ is $x = r\cos\theta; y = r\sin\theta$

(ii) DERIVATIVES OF FUNCTIONS IN PARAMETRIC FORM

Let $x = u(\theta)$ and $y = v(\theta)$ be the parametric form of a function, then

(i) $$\frac{dy}{dx} = \frac{\left(\frac{dy}{d\theta}\right)}{\left(\frac{dx}{d\theta}\right)}$$

(ii) $$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \left[\frac{d}{d\theta}\left(\frac{dy}{dx}\right)\right]\left(\frac{d\theta}{dx}\right) = \frac{\left[\frac{d}{d\theta}\left(\frac{dy}{dx}\right)\right]}{\left(\frac{dx}{d\theta}\right)}$$

Very Important: $$\frac{d^2y}{dx^2} \neq \frac{\left(\frac{d^2y}{d\theta^2}\right)}{\left(\frac{d^2x}{d\theta^2}\right)}$$

Example: Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ for the function $x = a\cos\theta, y = a\sin\theta$

Solution: $\frac{dy}{d\theta} = a\cos\theta$; $\frac{dx}{d\theta} = -a\sin\theta$; $\therefore \frac{dy}{dx} = \frac{\left(\frac{dy}{d\theta}\right)}{\left(\frac{dx}{d\theta}\right)} = \frac{a\cos\theta}{-a\sin\theta} = -\cot\theta$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \left[\frac{d}{d\theta}\left(\frac{dy}{dx}\right)\right]\left(\frac{d\theta}{dx}\right) = \frac{\left[\frac{d}{d\theta}\left(\frac{dy}{dx}\right)\right]}{\left(\frac{dx}{d\theta}\right)} = \frac{\left[\frac{d}{d\theta}(-\cot\theta)\right]}{\left(\frac{dx}{d\theta}\right)} = \frac{\mathrm{cosec}^2\theta}{-a\sin\theta}$$

$$= -a\,\mathrm{cosec}^3\theta$$

9.5 SECOND ORDER DERIVATIVE

Let $y = f(x)$. Then $\frac{dy}{dx} = f'(x)$.

Here, $f'(x)$ is called the **First Order Derivative** of f w.r.t. x.

If $f'(x)$ is further differentiable, then the expression obtained by differentiating $f'(x)$ w.r.t. x is called the **Second Order Derivative** of f w.r.t. x, denoted as or $f''(x)$.

REMARK

(i) The second order derivative of f w.r.t. x is denoted as $\frac{d}{dx}\left(\frac{dy}{dx}\right)$ **or** $\frac{d^2y}{dx^2}$ **or** $f''(x)$ **or** D **or** y'' **or** y_2.

(ii) The higher order derivatives, namely, the third order derivative, fourth order derivative etc. may also be defined similarly.

Example: Find the second derivative of $f(x) = \tan x$

Solution: First derivative of $f = f'(x) = \frac{d}{dx}(\tan x) = \sec^2 x$,

Second derivative of $f = f''(x) = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}(\sec^2 x) = 2(sec\ x)(sec\ x.\tan x)$

$= 2\sec^2 x\,.\tan x$

(iii) Second Order Derivative of Parametric Functions

Refer Derivative of functions in Parametric Forms (9.4)