FORMULAE & KEY POINTS

EXPONENTIAL AND LOGARITHMIC FUNCTIONS

APPENDIX TO CH. 5. CONTINUITY AND DIFFERENTIABILITY

1. EXPONENTIAL FUNCTION
2. LOGARITHMIC FUNCTION

1. EXPONENTIAL FUNCTION

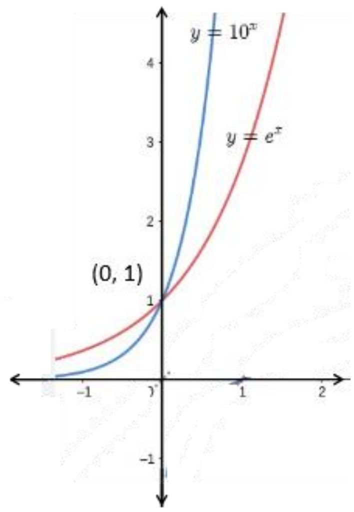
1.(i) Definition

The exponential function with positive base $a > 1$ is the function is defined as

$$y = f(x) = a^x$$

(ii) Domain $=$ R

(iii) Range $=$ R^+ or $(0,\infty)$



(iv) Graph of Exponential Function

REMARKS

(i) The point (0, 1) is always on the graph of the exponential function (as at $x = 0, y = a^0 = 1$ for any real $a > 1$).

(ii) Exponential function is ever increasing
⇔ as x increases(or decreases), y also increases(or decreases)
⇔as we move from left to right, the graph rises above

(iii) For very large negative values of x, the exponential function is very close to 0. Thus, in the second quadrant, the graph approaches x-axis (but never meets it). Mathematically, we say – the graph is **asymptotic** to x-axis

(iv) **Common Exponential Function:** Exponential function with base 10 is called the common exponential function.

(v) **Natural Exponential Function:** Exponential function with base e is called the natural exponential function.

Here, e is called Euler's constant defined as

$$e = \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + . . . = 2.718 \text{ (approx.)}$$

(vi) $\frac{d}{dx}(a^x) = a^x . \log_e a$

In particular, replacing a by e we get, $\frac{d}{dx}(e^x) = e^x$

2. LOGARITHMIC FUNCTION

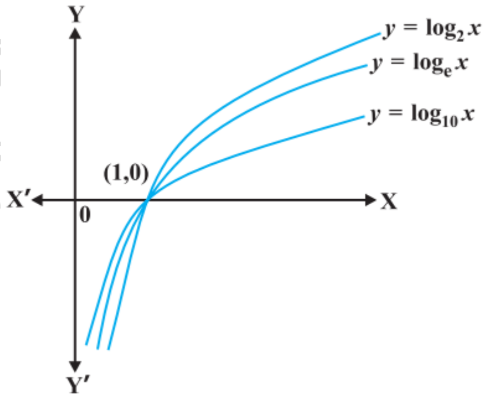
1.(i) **Definition:**

Let $a > 1$ be a real number. Then we say logarithm of x to the base a is y, if $a^y = x$.

Symbolically, $y = f(x) = \log_a x$ if $a^y = x$

(ii) **Domain** = R^+ or $(0,\infty)$

(iii) **Range** = R



(iv) **Graph of Logarithmic Function**

REMARKS

(i) The point (1, 0) is always on the graph of the logarithmic function (as at $x = 1, y = 0$ for any real $a > 1$).

(ii) Logarithmic function is ever increasing
⇔ as x increases(or decreases), y also increases(or decreases)
⇔as we move from left to right, the graph rises above

(iii) For x very near to zero, the value of log x can be made lesser than any given real number, the exponential function is very close to 0. Thus, in the fourth quadrant, the graph approaches y-axis (but never meets it). Mathematically, we say – the graph is **asymptotic** to y-axis

(iv) **Common Logarithmic Function:** Logarithmic function with base 10 is called the common logarithmic function.

(v) **Natural Logarithmic Function:** Logarithmic function with base e is called the natural logarithmic l function.

(vi) Logarithm with base e is written as $\ln x$

(vii) If no base is mentioned for a logarithmic function, it is assumed to be $\boldsymbol{e}$,

$$\text{i.e. } \log x = \log_e x$$

(viii) For any base a, we have the following results:

(a) As $a^0 = 1 \Rightarrow \log_a 1 = 0$

(b) As $a^1 = a \Rightarrow \log_a a = 1$

(c) As, $a^x = a^x \Rightarrow \log_a a^x = x$

(d) As, $\log_a x = \log_a x \Rightarrow a^{\log_a x} = x$

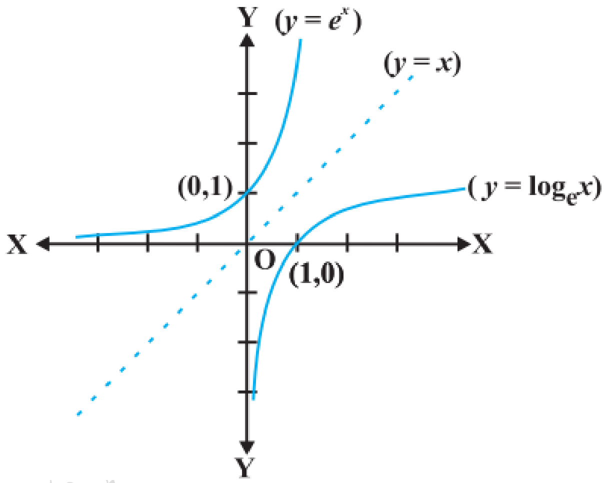
(ix) In view of (vii), replacing $\boldsymbol{a}$ by $\boldsymbol{e}$ in (viii) we get the following results:

(**a**) $\log 1 = 0$ (**b**) $\log e = 1$ (**c**) $\log e^x = x$ (**d**) $e^{\log x} = x$ (**e**) $e^{a \log x} = x^a$

(x) Exponential and Logarithmic functions are inverse of each other.

$$\text{i.e. } y = a^x \Leftrightarrow x = \log_a y$$

Thus, Exponential and Logarithmic functions are mirror images of each other with respect to the line $y = x$ as shown in the graph given below.



(xi) $\dfrac{d}{dx}(\log_a x) = \dfrac{1}{x \log a}$

In particular, replacing a by e we get, $\dfrac{d}{dx}(\log_e x) = \dfrac{1}{x}$

PROPERTIES OF LOGARITHMIC FUNCTIONS

For any base a, the following laws hold:

(i) $\log(m \times n) = \log m + \log n$ **(Product Law)**

Example

$\log[(x+1)(x+2)(x-3)] = \log(x+1) + \log(x+2) + \log(x-3)$

(ii) $\log\left(\dfrac{m}{n}\right) = \log m - \log n$ (**Quotient Law**)

Example

$\log\left(\dfrac{x}{\cos x}\right) = \log x - \log \cos x$

(iii) $\log m^n = n \log m$ **(Law of Exponent)**

Example 1

$\log y^x = x \log y$

Example 2

Using the above laws we can write $y = \sqrt{\dfrac{(x-2)^3(x+3)}{(2x-5)(x+4)(3x+7)}}$ as

$$\log y = \frac{1}{2}[3\log(x-2) + \log(x+3) - \log(2x-5) - \log(x+4) - \log(3x+7)]$$

(iv) $\log_a b = \dfrac{\log_c b}{\log_c a}$, where $c > 1$ (**Change of Base Law**)

REMARK

Replacing b by a above, we get

$\log_a b = \dfrac{1}{\log_b a} \Leftrightarrow (\log_a b) \times (\log_b a) = 1$ (**Reciprocal Law**)