FORMULAE & KEY POINTS

CLASS 12 MATHEMATICS

CH. 3. INVERSE TRIGONOMETRIC FUNCTIONS

1. PRINCIPAL VALUE BRANCH OF TRIGONOMETRIC FUNCTIONS

The restricted domain of a trigonometric function, lying in the interval $[-\pi, \pi]$, in which the trigonometric function is bijective (i.e. one - one and onto) is called the Principal Value Branch of that trigonometric function.

2.1 INVERSE TRIGONOMETRIC FUNCTIONS

The inverse trigonometric function $f(x) = \sin^{-1}x$ is defined as:

$$f(x) = \sin^{-1}x = \theta$$

where θ is the angle lying in the Principal Value Branch such that $\sin\theta = x$
The other inverse trigonometric function $\cos^{-1}x$, $\tan^{-1}x$, , $\cot^{-1}x$, $\sec^{-1}x$ and $\mathrm{cosec}^{-1}x$ are defined in a similar way.

REMARKS

(i) $\sin^{-1}x$ should not be confused with $(\sin x)^{-1}$. In fact $(\sin x)^{-1} = \dfrac{1}{\sin x}$ and similarly for other trigonometric functions.

(ii) Whenever no branch of an inverse trigonometric functions is mentioned, we mean the principal value branch of that function.

(iii) The value of an inverse trigonometric functions which lies in the range of principal branch is called the principal value of that inverse trigonometric functions

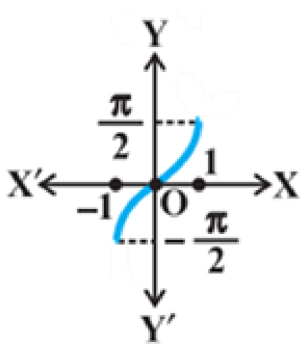
2.2 TRIGONOMETRIC RATIOS OF SOME SPECIFIC ANGLES

	Angle in Degrees (A) → Trigonometric Ratios ↓	0°	30°	45°	60°	90°
(i)	sin A	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
(ii)	cos A	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
(iii)	tan A	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	Not defined
(iv)	cot A	Not defined	$\sqrt{3}$	1	$\frac{1}{\sqrt{3}}$	0
(v)	sec A	1	$\frac{2}{\sqrt{3}}$	$\sqrt{2}$	2	Not defined
(vi)	cosec A	Not defined	2	$\sqrt{2}$	$\frac{2}{\sqrt{3}}$	1

2.3 DOMAIN AND RANGE OF INVERSE TRIGONOMETRIC FUNCTIONS

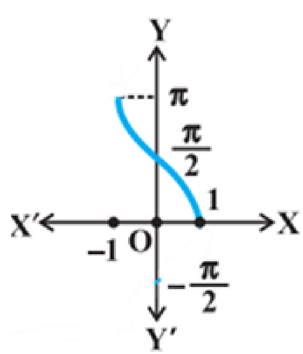
	Function $[y = f(x)]$	Domain (x)	Principal Value Branch/Range (y)
(i)	$\sin^{-1} x$	$[-1, 1]$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
(ii)	$\cos^{-1} x$	$[-1, 1]$	$[0, \pi]$
(iii)	$\tan^{-1} x$	R **or** $(-\infty, \infty)$	$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$
(iv)	$\cot^{-1} x$	R **or** $(-\infty, \infty)$	$(0, \pi)$
(v)	$\sec^{-1} x$	$(-\infty, -1] \cup [1, \infty)$ **or** $R - (-1,1)$	$[0, \pi] - \left\{\frac{\pi}{2}\right\}$ **or** $\left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$
(vi)	$\text{cosec}^{-1} x$	$(-\infty, -1] \cup [1, \infty)$ **or** $R - (-1,1)$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\}$ **or** $\left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$

2.4 GRAPHS OF INVERSE TRIGONOMETRIC FUNCTIONS



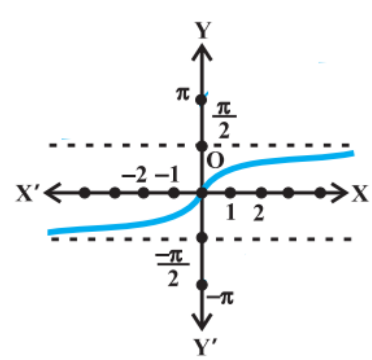
$y = \sin^{-1} x$

(i)



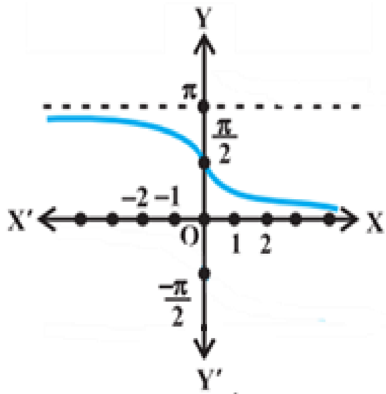
$y = \cos^{-1} x$

(ii)



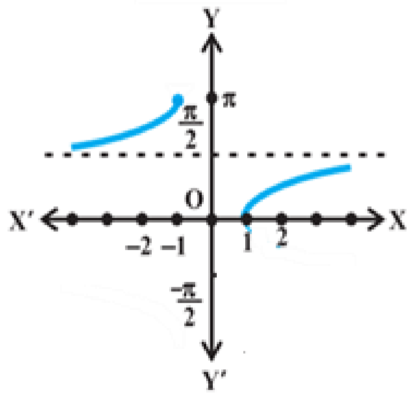
$y = \tan^{-1} x$

(iii)



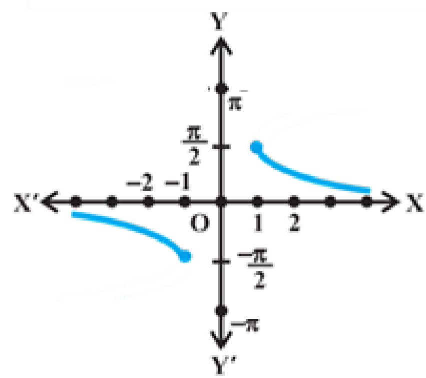
$y = \cot^{-1}x$

(iv)



$y = \sec x$

(v)



$y = \text{cosec}^{-1}x$

(vi)

REMARK

The graphs of inverse trigonometric functions are the mirror images of the respective trigonometric functions w.r.t. the line $y = x$.

3.1 If x lies in the respective domain, then

$$\sin(\sin^{-1}x) = x\,;\ \ \cos(\cos^{-1}x) = x \ \text{ and so on.}$$

3.2 If x lies in the Principal Value Branch, then

$$\sin^{-1}(\sin x) = x\,;\ \ \cos^{-1}(\cos x) = x \ \text{ and so on.}$$

EXAMPLES

(i) $\sin^{-1}\left[\sin\left(\frac{16\pi}{3}\right)\right] = \sin^{-1}\left[\sin\left(5\pi + \frac{\pi}{3}\right)\right] = \sin^{-1}\left[-\sin\left(\frac{\pi}{3}\right)\right] = -\sin^{-1}\left[\sin\left(\frac{\pi}{3}\right)\right] = -\frac{\pi}{3}$

(ii) $\sin^{-1}\left[\sin\left(\frac{19\pi}{4}\right)\right] = \sin^{-1}\left[\sin\left(5\pi - \frac{\pi}{4}\right)\right] = \sin^{-1}\left[\sin\left(\frac{\pi}{4}\right)\right] = \frac{\pi}{4}$

REMARKS

(i) $\sin^{-1}(\cos x) = \sin^{-1}\left[\sin\left(\frac{\pi}{2} - x\right)\right] = \frac{\pi}{2} - x\,.$

(ii) $\tan^{-1}(\cot x) = \tan^{-1}\left[\tan\left(\frac{\pi}{2} - x\right)\right] = \frac{\pi}{2} - x$

(iii) $\sec^{-1}(\text{cosec}\, x) = \sec^{-1}\left[\sec\left(\frac{\pi}{2} - x\right)\right] = \frac{\pi}{2} - x$

Similary we can simplify $\cos^{-1}(\sin x), \cot^{-1}(\tan x)$ and $\text{cosec}^{-1}(\sec x)$

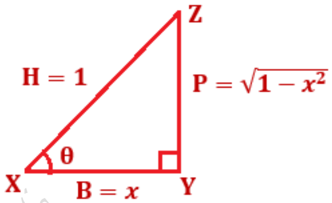
3.3(i) $\sin(\cos^{-1}x) = \sin\left(\sin^{-1}\sqrt{1-x^2}\right) = \sqrt{1-x^2}$

TRICK:

Let, $\theta = \cos^{-1}x = \cos^{-1}\left(\frac{x}{1}\right)$

Since $\cos\theta = \frac{B}{H}$, consider a right ΔXYZ whose Base is x and Hypotenuse is 1.

Then by Pythagoras theorem, Perpendicular, P = $\sqrt{1-x^2}$.



Hence, $\theta = \sin^{-1}\frac{\sqrt{1-x^2}}{1} = \sin^{-1}\sqrt{1-x^2}$

Thus, $\sin\left(\sin^{-1}\sqrt{1-x^2}\right) = \sqrt{1-x^2}$

(ii) $\sin(\tan^{-1}x) = \sin\left(\sin^{-1}\frac{x}{\sqrt{1+x^2}}\right) = \frac{x}{\sqrt{1+x^2}}$

(iii) $\sin(\cot^{-1}x) = \sin\left(\sin^{-1}\frac{\sqrt{1+x^2}}{x}\right) = \frac{\sqrt{1+x^2}}{x}$

(iv) $\sin(\sec^{-1}x) = \sin\left(\sin^{-1}\frac{\sqrt{x^2-1}}{x}\right) = \frac{\sqrt{x^2-1}}{x}$

(v) $\sin(\mathrm{cosec}^{-1}x) = \sin\left(\sin^{-1}\frac{1}{x}\right) = \frac{1}{x}$

Similar results hold for other trigonometric functions.

VERY IMPORTANT:

For convenience, while writing angle in the form $(n\pi \pm \theta)$, we choose the sign + or − in such a way that the angle θ becomes acute angle.

Consequently, in example (i), $\left(\frac{16\pi}{3}\right)$ has been written as $\left(5\pi + \frac{\pi}{3}\right)$ and not as $\left(6\pi - \frac{2\pi}{3}\right)$ whereas in example (ii), $\left(\frac{19\pi}{4}\right)$ has been written as $\left(5\pi - \frac{\pi}{4}\right)$ and not as $\left(4\pi + \frac{3\pi}{4}\right)$

4(i) $\sin^{-1}\left(\frac{1}{x}\right) = \mathrm{cosec}^{-1}x, \quad x \ge 1 \text{ or } x \le -1$

(ii) $\cos^{-1}\left(\frac{1}{x}\right) = \sec^{-1}x, \quad x \ge 1 \text{ or } x \le -1$

(iii) $\tan^{-1}\left(\frac{1}{x}\right) = \cot^{-1}x, \quad x > 0$

(iv) $\cot^{-1}\left(\frac{1}{x}\right) = \tan^{-1}x, \quad x > 0$

(v) $\sec^{-1}\left(\frac{1}{x}\right) = \cos^{-1}x, \quad |x| \le 1$

(vi) $\mathrm{cosec}^{-1}\left(\frac{1}{x}\right) = \sin^{-1}x, \quad |x| \le 1$

5(i) $\sin^{-1}(-x) = -\sin^{-1}x, \qquad x \in [-1, 1]$

(ii) $\cos^{-1}(-x) = \pi - \cos^{-1}x, \qquad x \in [-1, 1]$

(iii) $\tan^{-1}(-x) = -\cot^{-1}x, \qquad x \in \mathrm{R}$

(iv) $\cot^{-1}(-x) = \pi - \cos^{-1}x, \qquad x \in \mathrm{R}$

(v) $\sec^{-1}(-x) = \pi - \sin^{-1}x, \qquad |\,x\,| \geq 1$

(vi) $\mathrm{cosec}^{-1}(-x) = -\mathrm{cosec}^{-1}x, \qquad |\,x\,| \geq 1$

REMARK

The inverse trigonometric functions like $\cos^{-1}x$ whose Principal Value Branches are associated with the interval $(0, \pi)$ have formulae of the type $(\pi - \cos^{-1}x)$.

6(i) $\sin^{-1}x + \cos^{-1}x = \dfrac{\pi}{2}, \qquad x \in [-1,1]$

(ii) $\tan^{-1}x + \cot^{-1}x = \dfrac{\pi}{2}, \qquad x \in \mathrm{R}$

(iii) $\sec^{-1}x + \mathrm{cosec}^{-1}x = \dfrac{\pi}{2}, \qquad |x| \geq 1$

7. TIPS FOR WRITING THE EXPRESSIONS INVOLVING INVERSE TRIGONOMETRIC FUNCTIONS IN SIMPLE FORM

7.1	Expression	Substitution		Expression	Substitution
(i)	$\sqrt{1 - x^2}$	$x = \sin\theta$ **or** $\cos\theta$	(iv)	$\sqrt{a^2 - x^2}$	$x = a\sin\theta$ **or** $a\cos\theta$
(ii)	$\sqrt{1 + x^2}$	$x = \tan\theta$ **or** $\cot\theta$	(v)	$\sqrt{a^2 + x^2}$	$x = a\tan\theta$ **or** $a\cot\theta$
(iii)	$\sqrt{x^2 - 1}$	$x = \sec\theta$ **or** $\mathrm{cosec}\,\theta$	(vi)	$\sqrt{x^2 - a^2}$	$x = a\sec\theta$ **or** $a\,\mathrm{cosec}\,\theta$

7.2(i) $1 \pm 2\sin x. = \sin^2\frac{x}{2} + \sin^2\frac{x}{2} \pm 2\sin\frac{x}{2}\cos\frac{x}{2} = \left(\sin\frac{x}{2} \pm \cos\frac{x}{2}\right)^2$

(ii)
$$\sqrt{1 - 2\sin 2x} = \begin{cases} \cos x - \sin x \text{ if } x \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right) \text{ as } (\cos x - \sin x) > 0 \text{ for } x \in \left(0, \frac{\pi}{4}\right) \\ \sin x - \cos x \text{ if } x \in \left(0, \frac{\pi}{4}\right) \text{ as } (\sin x - \cos x) > 0 \text{ for } x \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right) \end{cases}$$

(iii)
$$\frac{\cos x \mp \sin x}{\cos x \pm \sin x} = \frac{1 \mp \tan x}{1 \pm \tan x}\,(by\ dividing\ N^r and\ D^r by\ cos\ x) = \frac{\tan\frac{\pi}{4} \mp \tan x}{1 \pm \tan\frac{\pi}{4}\tan x}\,(using\ \ tan\frac{\pi}{4} = 1)$$

$$= \tan\left(\frac{\pi}{4} \mp x\right) \left[\text{Using the formula } \tan(x \pm y) = \frac{\tan x \pm \tan y}{1 \mp \tan x\ \tan y}\right]$$

(iii)
$$\frac{\cos x}{1 \pm \sin x} = \frac{\cos^2\frac{x}{2} - \sin^2\frac{x}{2}}{\cos^2\frac{x}{2} + \sin^2\frac{x}{2} \pm 2\sin^2\frac{x}{2}.\cos\frac{x}{2}} \left[\text{Using } \cos 2x = \cos^2 x - \sin^2 x\ \ \&\sin^2 x + \cos^2 x = 1\right]$$

$$= \frac{\left(\cos\frac{x}{2} - \sin\frac{x}{2}\right)\left(\cos\frac{x}{2} + \sin\frac{x}{2}\right)}{\left(\cos\frac{x}{2} \pm \sin\frac{x}{2}\right)^2} [Using\ a^2 - b^2 = (a+b)(a-b)\ \&\ a^2 + b^2 \pm 2ab = (a \pm b)^2]$$

$$= \frac{\cos\frac{x}{2} \mp \sin\frac{x}{2}}{\cos\frac{x}{2} \pm \sin\frac{x}{2}} [\text{cancelling common terms from N}^r \text{and D}^r]$$

Proceeding as in (ii) we get

$$\frac{1 \mp \tan\frac{x}{2}}{1 \pm \tan\frac{x}{2}} = \frac{\tan\frac{\pi}{4} \mp \tan\frac{x}{2}}{1 \pm \tan\frac{\pi}{4}.\tan\frac{x}{2}} = \tan\left(\frac{\pi}{4} \mp \frac{x}{2}\right)$$

THE FOLLOWING PROPERTIES ARE NOT IN C.B.S.E. SYLLABUS CLASS 12 SYLLABUS

8.1(i) $2\sin^{-1}x = \sin^{-1}(x\sqrt{1-x^2});\quad -\frac{1}{\sqrt{2}} \le x \le \frac{1}{\sqrt{2}}$

(ii) $2\ cos^{-1}x = \cos^{-1}(2x^2 - 1);\ \ 0 \le x \le 1$

(iii) $2\tan^{-1}x\ =\ \tan^{-1}\left(\frac{2x}{1-x^2}\right);\ \ |x| < 1$

(iv) $2\tan^{-1}x\ =\ \sin^{-1}\left(\frac{2x}{1+x^2}\right);\ \ |x| \le 1$

(v) $2\tan^{-1}x\ =\ \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right);\ \ x \ge 0$

8.2(i) $$\sin^{-1}x + \sin^{-1}y = \begin{cases} \sin^{-1}(x\sqrt{1-y^2} + y\sqrt{1-x^2});\text{if } x, y \in [-1,1] \text{ and } x^2+y^2 \le 1 \\ \qquad\qquad \textbf{or}\ \text{xy} < 0 \text{ and } x^2+y^2 > 1 \\ \pi - \sin^{-1}(x\sqrt{1-y^2} + y\sqrt{1-x^2});\text{if } x, y \in (0,1] \text{ and } x^2+y^2 > 1 \\ -\pi - \sin^{-1}(x\sqrt{1-y^2} + y\sqrt{1-x^2});\text{if } x, y \in [-1,0) \text{ and } x^2+y^2 > 1 \end{cases}$$

(ii) $$\sin^{-1}x - \sin^{-1}y = \begin{cases} \sin^{-1}\left(x\sqrt{1-y^2} - y\sqrt{1-x^2}\right);\text{if } x, y \in [-1,1] \text{ and } x^2+y^2 \le 1 \\ \qquad\qquad \textbf{or}\ \text{xy} > 0 \text{ and } x^2+y^2 > 1 \\ \pi - \sin^{-1}\left(x\sqrt{1-y^2} - y\sqrt{1-x^2}\right);\text{if } x \in (0,1], \text{y} \in [-1,0) \text{ and } x^2+y^2 > 1 \\ -\pi - \sin^{-1}\left(x\sqrt{1-y^2} - y\sqrt{1-x^2}\right); x \in [-1,0), \text{y} \in (0,1] \text{ and } x^2+y^2 > 1 \end{cases}$$

(iii) $$\cos^{-1}x + \cos^{-1}y = \begin{cases} \cos^{-1}\left(xy - \sqrt{1-y^2}\sqrt{1-x^2}\right);\ \text{if } x, y \in [-1,1] \text{ and } x+y \ge 0 \\ 2\pi - cos^{-1}\left(xy - \sqrt{1-y^2}\sqrt{1-x^2}\right);\ \text{if } x, y \in [-1,1] \text{ and } x+y \le 0 \end{cases}$$

(iv) $$\cos^{-1}x - \cos^{-1}y = \begin{cases} \cos^{-1}\left(xy + \sqrt{1-y^2}\sqrt{1-x^2}\right);\ \text{if } x, y \in [-1,1] \text{ and } x \le y \\ -cos^{-1}\left(xy + \sqrt{1-y^2}\sqrt{1-x^2}\right);\ \text{if } x \in [-1,0), \text{y} \in (0,1] \text{ and } x \ge y \end{cases}$$

(v) $$\tan^{-1}x + \tan^{-1}y = \begin{cases} \tan^{-1}\left(\dfrac{x+y}{1-xy}\right);\ \text{if } xy < 1 \\ \pi + \tan^{-1}\left(\dfrac{x+y}{1-xy}\right);\ \text{if } x > 0, y > 0 \text{ and } xy > 1 \\ -\pi + \tan^{-1}\left(\dfrac{x+y}{1-xy}\right);\ \text{if } x < 0, y < 0 \text{ and } xy > 1 \end{cases}$$

(vi) $$\tan^{-1}x - \tan^{-1}y = \begin{cases} \tan^{-1}\left(\dfrac{x-y}{1+xy}\right);\ \text{if xy} > -1 \\ \pi + \tan^{-1}\left(\dfrac{x-y}{1+xy}\right);\ \text{if } x > 0, y < 0 \text{ and } xy < -1 \\ -\pi + \tan^{-1}\left(\dfrac{x-y}{1+xy}\right);\ \text{if } x < 0, y > 0 \text{ and } xy < -1 \end{cases}$$