FORMULAE & KEY POINTS
CLASS 12 MATHEMATICS
CHAPTER 01 : RELATIONS AND FUNCTIONS

1. RELATION

1.1 DEFINITION OF RELATION

A relation R from a non-empty set A to a non − empty set B is a subset of the cartesian product A × B.

1.2 IMAGE, PRE-IMAGE, DOMAIN, CO-DOMAIN AND RANGE IN A RELATION

Let R A → B be a relation from non − empty set A to another non − empty set B.
Then, R = $\{(x, y) x \in A$ and $y \in B;\ a\ R\ b\}$ and we have the following

(i) Image and Pre-Image

In an ordered pair (x, y) belonging to the relation R,
(**a**)the element y is called the **Image** of x under the relation R and
(**b**)the element x is called the **Pre-Image** of y under the relation R

(ii) Domain

The set of all first elements of the ordered pairs (x, y)in a relation R is called the domain of the relation R.
Symbolically, Domain = $\{x (x, y) \epsilon R\}$

(iii) Co-Domain

The entire set B is called the codomain of the relation R.

(iv) Range

The set of all second elements of the ordered pairs (x, y) is called the range of the relation R.
Symbolically, Range = $\{y (x, y) \epsilon R\}$

REMARKS

(i) Range ⊂ Codomain

(ii) A relation may be represented in the following ways
(**A**) in Roster form (**B**) in Set-builder form (**C**) in Arrow Diagram form

2. TYPES OF RELATIONS

2.1 EMPTY RELATION

A relation R in a set A is called empty relation, if no element of A is related to any element of A, i.e., $R = \emptyset \subset A \times A$.

2.2 UNIVERSAL RELATION

A relation R in a set A is called universal relation, if each element of A is related to every element of A, i.e., $R = A \times A$

REMARKS

(i) Both the empty relation and the universal relation are sometimes called **Trivial Relations.**.

(ii) If $(a, b) \in R$, we say that a is related to b and we denote it as a R b

2.3 REFLEXIVE, SYMMETRIC AND TRANSITIVE RELATIONS

A relation R in a set A is called

(i) **Reflexive**, if $(a, a) \in R$, for every $a \in A$,

(ii) **Symmetric**, if $(a, b) \in R$ implies that $(b, a) \in R$, for all $a, b \in A$.

(iii) **Transitive**, if $(a, b) \in R$ and $(b, c) \in R$ implies that $(a, c) \in R$, for all $a, b, c \in A$

VERY IMPORTANT REMARK

If in a relation R, there are no elements of the type (a, b) and (b, c) to check transitivity. Then by default we consider R to be transitive.

(Refer Example 1 given below)

Example 1

ısider the relation R in the set of human being defined as $\{(x, y): x$ is the wife of $y\}$. Then R is a transitive relation though for an element $(x, y) \in R$ the element of the type (y, z) can not never belong to R.

Example 2

Let L be the set of all lines in XY plane and R be the relation in L defined as

$R = \{(l, m)\ l$ is parallel to $m\}$. Then

(i) R is reflexive since every line is parallel to itself $i.e.$ For every $l \in L, l \parallel l$

(ii) R is symmetric since if a line l is parallel to another line m, then the second line m is parallel to the first line l $i.e.$ For $l, m \in L, l \parallel m \Rightarrow m \parallel l$.

(iii) R is transitive since if a line l is parallel to a second line m, and m is parallel to the line line n, then the first line l is also parallel to the third line n. $i.e.$ For $l, m, n \in L$, $l \parallel m$ and $m \parallel n \Rightarrow l \parallel n$.

Example 3

Let L be the set of all lines in XY plane and R be the relation in L defined as
R $= \{(l, m)\ l$ is perpendicular to $m\}$. Then

(i) R is not reflexive since no line is perpendicular to itself $i.e.\ l \perp l$ *is* false for every $l \in$ L

(ii) R is symmetric since if a line l is perpendicular to another line m, then the second line m is also perpendicular to the first line l, that is, for $l, m \in$ L, $l \perp m \Rightarrow m \perp l$.

(iii) R is not transitive since if a line l is perpendicular to a second line m, and m is perpendicular to a third line n, then the first line l is not perpendicular to the third line n, that is, for $l, m, n \in$ L, $l \perp m$ and $m \perp n \nRightarrow l \perp n$ (In fact, in this case, $l \parallel n$).

2.4 EQUIVALENCE RELATION

A relation R in a set A is said to be an equivalence relation if R is reflexive, symmetric and transitive.

Example

The Relation given in Example 1 above is an equivalence relation.

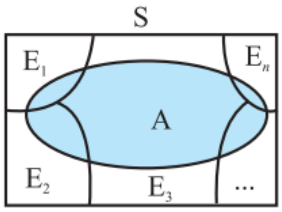
3. PARTITION OF A SET

3.1 Definition

Let S be a non-empty set. A collection of subsets of S namely $\mathrm{E}_1, \mathrm{E}_2, \mathrm{E}_3, \ldots, \mathrm{E}_n$ is said to be a partition of S if

(i) $\cup\, \mathrm{E}_j = \mathrm{S}$ and

(ii) $\mathrm{E}_i \cap \mathrm{E}_j = \phi, i \neq j$



3.2 Partition of a Set Formed by an Equivalent Relation

Given an arbitrary equivalence relation R in an arbitrary set X, R divides X into mutually disjoint subsets E_i called partitions or subdivisions of X satisfying

(i) all elements of E_i are related to each other, for all i.

(ii) no element of E_i is related to any element of E_j, $i \neq j$.

(iii) $\cup\, \mathrm{E}_j = \mathrm{X}$ and $\mathrm{E}_i \cap \mathrm{E}_j = \phi, i \neq j$

The subsets E_i are called **Equivalence Classes**.

Example

Consider the equivalence relation R in the set Z of integers given by
R $= \{(\mathrm{a}, \mathrm{b})\ 2$ divides $\mathrm{a} - \mathrm{b}\}$. Then we have the following implications:

- all even integers are related to zero, as $(0, \pm 2), (0, \pm 4)$ etc., lie in R and
- no odd integer is related to 0, as $(0, \pm 1), (0, \pm 3)$etc., do not lie in R.
- similarly, all odd integers are related to one and no even integer is related to one.

Therefore, the set E of all even integers and the set O of all odd integers are said to form a partition of X.
The subset E is called the equivalence class containing zero and is denoted by [0].
Similarly, the subset O is the equivalence class containing 1 and is denoted by [1].

REMARK

Every Partition of a set can give rise to an Equivalence Relation in the set.

Example

Consider a subdivision of the set Z given by three mutually disjoint subsets E_0 , E_1 and E_2 whose union is Z with

$E_0 = \{x \in Z\ x \text{ is a multiple of } 3\} = \{\ldots, -6, -3, 0, 3, 6, \ldots\}$

$E_1 = \{x \in Z\ x - 1 \text{ is a multiple of } 3\} = \{\ldots, -5, -2, 1, 4, 7,.$

$E_2 = \{x \in Z\ x - 2 \text{ is a multiple of } 3\} = \{\ldots, -4, -1, 2, 5, 8, \ldots\}$

Define a relation R in Z given by $R = \{(a, b)\ 3 \text{ divides } a - b\}$.

R is an equivalence relation.

Also, E_0 coincides with the set of all integers in Z which are related to zero,

E_1 coincides with the set of all integers which are related to 1 and

E_2 coincides with the set of all integers in Z which are related to 2.

Thus, $E_0 = [0], E_1 = [1]$ and $E_2 = [2]$.

In fact, $E_0 = [3r], E_1 = [3r + 1]$ and $E_2 = [3r + 2]$, for all $r \in Z$.

4. SOME FORMULAE FOR HIGH ACHIEVERS

4.1 Let, number of elements in two sets A and B are given as $n(A) = p$ and $n(B) = q$, then Number of relations from A to B = Number of subsets of $A \times B = 2^{p \times q}$

4.2 Let $n(A) = n$ then,

(i) Number of relations from A to A = Number of subsets of $A \times A = 2^{n^2}$

(ii) Number of reflexive relation from A to A $= 2^{n(n-1)} = 2^{n^2-n}$

(iii) Number of symmetric relation from A to A $= 2^{\frac{n(n+1)}{2}}$

(iv) Number of relation from A to A which are not symmetric

= (Number of relation from A to A) − (Number of reflexive relation from A to A)

$$= 2^{n^2} - 2^{\frac{n(n+1)}{2}}$$

5. FUNCTIONS

5.1 Definition

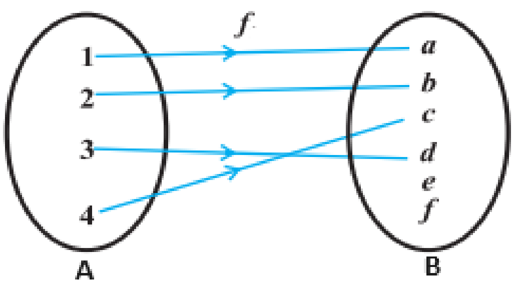
A function f from a non-empty set A to a non-empty set B is a relation from A to B which relates EACH element of A to a UNIQUE(*i.e.* one and only one) element of B.

6. TYPES OF FUNCTIONS

6.1 ONE-ONE OR INJECTIVE FUNCTION

6.1.1 Definition

A function $f A \to B$ is said to be One − One (or Injective) function , if the images of distinct elements of A under f are distinct.



6.1.2 Method to prove that a function $f: A \to B$ is One-One (or Injective) Function)

To prove a function f to be injective, we use ANY ONE of the following methods

(i) prove that for every $x_1, x_2 \in A, f(x_1) = f(x_2) \Rightarrow x_1 = x_2$.

(ii) prove that for every $x_1, x_2 \in A, x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$.

(iii) Horizontal Line Test

- Draw the graph of the function.
- Imagine horizontal lines at various y-values across the graph.
- If no horizontal line touches the graph at more than one point, the function is a one-one function.

6.1.3 Methods to prove that a function f is NOT One-One (or *NOT* Injective)

To prove a function f is not injective, we use ANY ONE of the following methods

(i) Give a counter example

Show that there exist elements x_1 and x_2 in the domain satisfying $x_1 \neq x_2$ such that $f(x_1) = f(x_2)$ [*i.e.* there exist two distinct elements in the domain having same image under f]

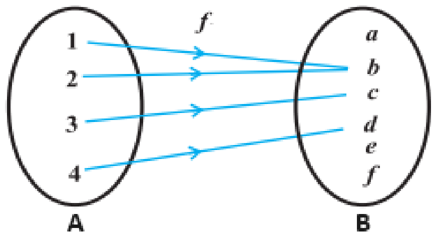
(ii) Horizontal Line Test

- Draw the graph of the function.
- Imagine horizontal lines at various y-values across the graph.
- If there exists at least one horizontal that line touches the graph at more than one point, the function is a not one-one function.

6.2 MANY-ONE FUNCTION

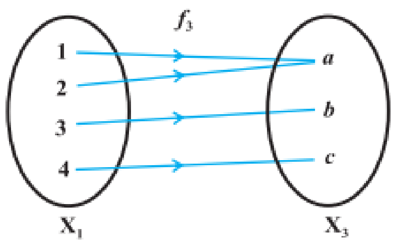
A function f A $\to$ B is said to be many-one if at least two elements of the domain A have the same image in the codomain B.

Thus, a function f A $\to$ B is said to be Many-One if f is not a one-one function.

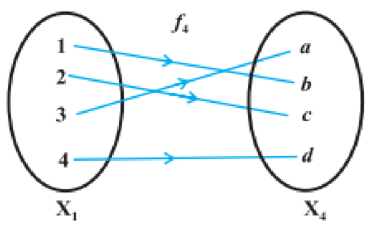


6.3 ONTO (OR SURJECTIVE) FUNCTION

6.3.1 Definition



Many -One Onto



One-One Onto

A function f A $\to$ B is said to be onto (or surjective), if every element of the codomain B is the image of some element of the domain A under f, i.e., for every $y \in$ codomain B, there exists an element x in the domain A such that $f(x) = y$

6.3.2 Methods to prove that a function $f A \rightarrow B$ is Onto (or Surjective)

To prove a function f is surjective, we use ANY ONE of the following methods

(i) Take a general element $y \in B$ (codomain)and show that corresponding to the elemeny y there exists an element $x \in A$ (domain) such that $f(x) = y$.

(ii) Graphical Method

- Draw the graph of the function.
- If every possible y-value in the codomain B has a corresponding point on the graph then the function is **onto.**

6.3.3 Method to prove that a function $f: A \rightarrow B$ is NOT Onto (or NOT Surjective)

To prove a function f is not surjective, we use ANY ONE of the following methods

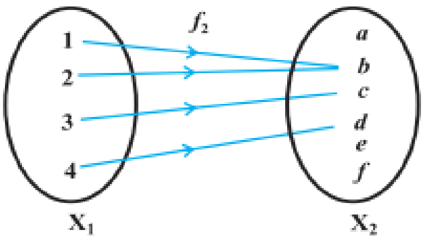
(i) Give a counter example

Show that there exists an element y in the codomain B which is not the image of any element x of the domain A [$i.e.$ there is no element $x \in A$ such that $f(x) = y$].

(ii) Graphical Method

- Draw the graph of the function.
- If there is a y-value in the codomain B for which no point exists on the graph then the function is not onto.

6.4 INTO FUNCTION



A function $f \ A \rightarrow B$ is said to be an into function, there is at least one element y in the codomain B which is not the image of any element of A.
Thus, a function $f \ A \rightarrow B$ is said to be an into function if it is not onto

7. SOME FORMULAE FOR HIGH ACHIEVERS

Let $f \ A \rightarrow B$ be a function where $n(A) = m$ and $n(B) = n$. Then,

(i) Total number of functions from A to B = n^m

$= \text{(Number of elements in codomain)}^{\text{Number.of elements in domain}}$

(ii) Total number of one-one functions from A to B = $\begin{cases} {}^{n}P_{r}, & \text{if } n \geq m \\ 0, & \text{if } n < m \end{cases}$

REMARK

If $n < m$, it is not possible to have one-one functions, so the number of one-one functions is 0.

(iii) Total number of many-one functions from A to B = $\begin{cases} n^m - {}^{n}P_{r}, & \text{if } n \geq m \\ n^m, & \text{if } n < m \end{cases}$

(iv) Total number of onto functions from A to B

$$\begin{cases} n^m - {}^{n}C_1 . (n-1)^m + {}^{n}C_2 . (n-2)^m - {}^{n}C_3 . (n-3)^m + \ldots & , \text{if } n < m \\ m! & , \text{if } n = m \\ 0 & , \text{if } n > m \end{cases}$$

REMARK

If $m < n$, it is not possible to have onto functions, so the number of onto functions is 0.

(v) If $n = m$, then total number of one-one and onto ($i.e.$ bijective) functions = $m!$