गणितालय GANITALAY गणितालय

FORMULAE & KEY POINTS

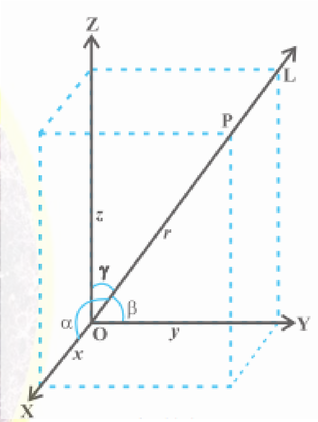
CLASS 12 MATHEMATICS

CHAPTER 11 : THREE DIMENSIONAL GEOMETRY

1. DIRECTION ANGLES, DIRECTION COSINES(D.C.[S]) & DIRECTION RATIOS(D.R.[S]) OF A DIRECTED LINE

1.1 DIRECTION ANGLED OF A DIRECTED LINE

The angles α, β and γ made by a directed line with the positive direction of $\boldsymbol{x}, \boldsymbol{y},$ and $\boldsymbol{z}$-axes respectively are called the **Direction Angles** of the directed line.



1.2 DIRECTION COSINES(D.C.[S]) OF A DIRECTED LINE

If α, β and γ are direction angles of a line then the quantities

$$\boldsymbol{l} = \cos\boldsymbol{\alpha}, \quad \boldsymbol{m} = \cos\boldsymbol{\beta}, \quad \boldsymbol{n} = \cos\boldsymbol{\gamma}$$

respectively are called the Direction Cosines of the line.

VERY IMPORTANT REMARKS

(i) $\boldsymbol{l^2 + m^2 + n^2 = 1}$.

(ii) A given line in space can be extended in two opposite directions and so it has two set of direction angles $i.e.$ α, β, γ and $\pi-\alpha, \pi-\beta, \pi-\gamma$.
Consequently, for a line there exist two sets of direction cosines: $\cos\alpha$, $\cos\beta$, $\cos\gamma$ and $\cos(\pi-\alpha) = -\cos\alpha$, $\cos(\pi-\beta) = -\cos\beta$, $\cos(\pi-\gamma) = -\cos\gamma$
$i.e.$ l, m, n and $-l$, $-m$, $-n$.
Thus, in order to have a unique set of direction cosines of a given line in space, we must take given line as a directed line.

1.3 DIRECTION RATIOS(D.R.[S]) OF A DIRECTED LINE

If l, m and n are the direction cosines of a directed line then for any non-zero real number k, the quantities $a = kl$, $b = km$ and $c = kn$ are called direction ratios of the line.

REMARKS

(i) The direction cosines of a directed line are unique whereas the direction ratios are not so.

(ii) Relation between direction cosines and direction ratios of a directed line:

$$l = \frac{a}{\sqrt{a^2+b^2+c^2}}, m = \frac{b}{\sqrt{a^2+b^2+c^2}}, n = \frac{c}{\sqrt{a^2+b^2+c^2}}$$

(iii) Relation between direction cosines and direction ratios of a non-directed line:

$$l = \pm\frac{a}{\sqrt{a^2+b^2+c^2}}, m = \pm\frac{b}{\sqrt{a^2+b^2+c^2}}, n = \pm\frac{c}{\sqrt{a^2+b^2+c^2}}$$

(iv) The direction cosines of a line are also its direction ratios (because for $k = 1, a = l, b = m$ and $c = n$)

(v) If a, b, c are the direction ratios of a line then for any non $-$ zero real number k, then ka, kb, kc are also the direction ratios of the line.

2. D.C.^{S} AND D.R.^{S} OF A LINE PASSING THROUGH TWO POINTS

For a line passing through the points $\mathrm{P}(x_1, y_1, z_1)$ and $\mathrm{Q}(x_2, y_2, z_2)$ and directed from P to Q,

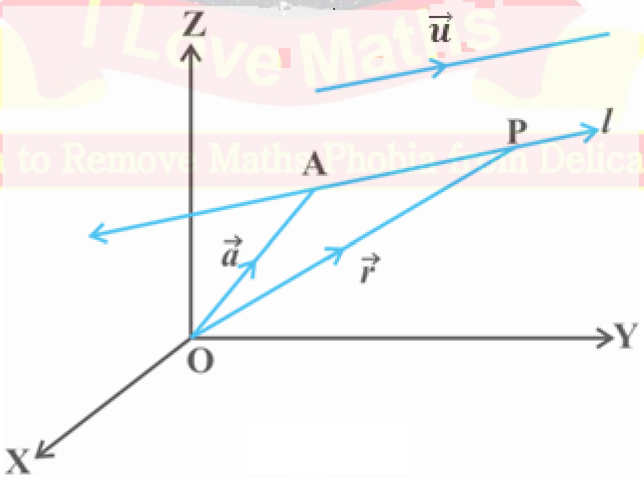
(i) The Direction Ratios are : $a = (x_2 - x_1)$, $b = (y_2 - y_1)$, $c = (z_2 - z_1)$ and

(ii) The Direction Cosines are: $l = \dfrac{x_2 - x_1}{\mathrm{PQ}}$, $m = \dfrac{y_2 - y_1}{\mathrm{PQ}}$, $n = \dfrac{z_2 - z_1}{\mathrm{PQ}}$

where, $\mathrm{PQ} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$ = length of the line segment PQ

3. EQUATION OF A LINE IN SPACE

3.1 EQUATION OF THE LINE PASSING THROUGH A POINT AND PARALLEL TO A GIVEN VECTOR



Equation of a Line pssing through the point $A(x_1, y_1, z_1)$ with position vector $\vec{p} = x_1\hat{\imath} + y_1\hat{\jmath} + z_1\hat{k}$ and parallel to the vector $\vec{u} = a_1\hat{\imath} + b_1\hat{\jmath} + c_1\hat{k}$:

Vector Equation : $\vec{r} - \vec{p} = \lambda\vec{u}$ **or** $\vec{r} = \vec{p} + \lambda\vec{u}$, $\lambda \in \mathrm{R}$

Cartesian Equation : $\dfrac{x - x_1}{a_1} = \dfrac{y - y_1}{b_1} = \dfrac{z - z_1}{c_1}$ $(= \lambda$, say)

REMARKS

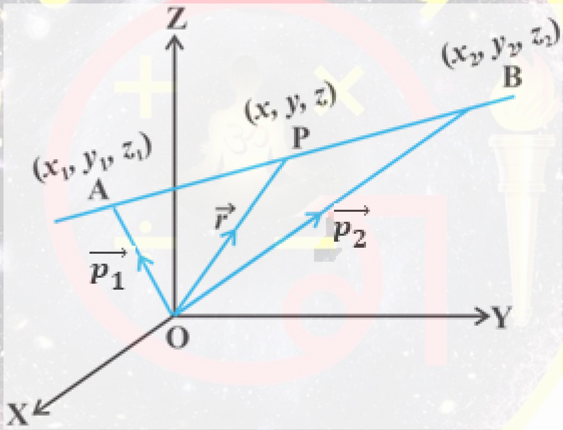
(i) The coordinates of a General Point on the above line is
$(\vec{a} + \lambda\vec{u})$ **OR** $P(x_1 + \lambda a_1, y_1 + \lambda b_1, z_1 + \lambda c_1)$

(iii) **Equations of x, y and z - axis**

	Axis	Vector Equation	Cartesian Equation
(a)	x-axis	$\vec{r} = \lambda\hat{\imath}$	$\frac{x}{1} = \frac{y}{0} = \frac{z}{0}$
(b)	y-axis	$\vec{r} = \lambda\hat{\jmath}$	$\frac{x}{0} = \frac{y}{1} = \frac{z}{0}$
(c)	z-axis	$\vec{r} = \lambda\hat{k}$	$\frac{x}{0} = \frac{y}{0} = \frac{z}{1}$

(iv) A point on x-axis can be taken as $A(a, 0, 0)$, a point on y-axis can be taken as $B(0, b, 0)$ and a point on z-axis can be taken as $C(0, 0, c)$

3.2 EQUATION OF THE LINE PASSING THROUGH TWO GIVEN POINTS



Equation of a Line passing through two point $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ with position vector $\vec{p_1} = x_1\hat{\imath} + y_1\hat{\jmath} + z_1\hat{k}$ and $\vec{p_2} = x_2\hat{\imath} + y_2\hat{\jmath} + z_2\hat{k}$:

Vector Equation: $\vec{r} = \vec{p_1} + \lambda(\vec{p_2} - \vec{p_1}),\ \lambda \in R$

Cartesian Equation: $\dfrac{x - x_1}{x_2 - x_1} = \dfrac{y - y_1}{y_2 - y_1} = \dfrac{z - z_1}{z_2 - z_1}$ $(= \lambda, \text{say})$

REMARKS

(i) To find the equation of the line passing through two points, any of the two points A and B can be taken as $\vec{p_1}$ or $\vec{p_2}$

(ii) The coordinates of a General Point on the above line is
$\vec{p_1} + \lambda(\vec{p_2} - \vec{p_1})$, **OR** $P(x_1 + \lambda(x_2 - x_1),\ y_1 + \lambda(y_2 - y_1),\ z_1 + \lambda(z_2 - z_1))$

4. ANGLE BETWEEN TWO LINES:

Let L_1 and L_2 be two lines passing through with direction ratios a_1, b_1, c_1 and a_2, b_2, c_2, respectively, then the acute angle θ between the lines L_1 and L_2 is given as

$$\cos\theta = \left|\frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}\right|$$

NOTE:

From above, $\sin\theta = \sqrt{1-\cos^2\theta} = \dfrac{\sqrt{(a_1b_2-a_2b_1)^2+(b_1c_2-b_2c_1)^2+(c_1a_2-c_2a_1)^2}}{\sqrt{a_1{}^2+b_1{}^2+c_1{}^2}\sqrt{a_2{}^2+b_2{}^2+c_2{}^2}}$

VERY IMPORTANT REMARK

Before using the equation of a given line, first ensure that the line is in standard form. If not, reduce it to standard form.

EXAMPLES

	Non-Standard Form of a line	Standard Form of the line
(i)	$\dfrac{3x+4}{5} = \dfrac{2-3y}{4} = z$	$\dfrac{x+4/3}{5/3} = \dfrac{y-2/3}{-4/3} = \dfrac{z-0}{1}$
(ii)	$\vec{r} = (2-3s)\hat{\imath} + (s-3)\hat{\jmath} + (2s+5)\hat{k}$	$\vec{r} = 2\hat{\imath} - 3\hat{\jmath} + 5\hat{k} + s(-3\hat{\imath} + \hat{\jmath} + 2\hat{k})$

5. PARALLEL AND PERPENDICULAR LINES

From point 4, the lines L_1 and L_2 are

(i) Parallel if $\sin\theta = 0 \iff \dfrac{a_1}{a_2} = \dfrac{b_1}{b_2} = \dfrac{c_1}{c_2}$

$\iff$ the Diretion Ratios of L_1 and L_2 are Proportional

(ii) Perpendicular if $\cos\theta = 0 \iff a_1a_2 + b_1b_2 + c_1c_2 = 0$

6. SKEW LINES

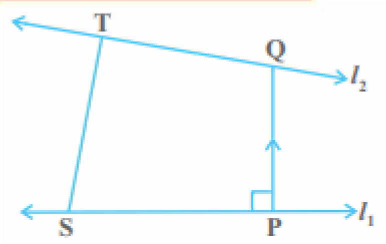
Two lines in space which are neither intersecting nor parallel are called as skew lines. These lines are non coplanar.

7. SHORTEST DISTANCE (OR SIMPLY DISTANCE)BETWEEN TWO SKEW LINES

7.1 VECTOR FORM

The shortest distance between two lines l_1 and l_2 whose vector equations are $\vec{r} = \vec{a}_1 + \lambda\,\vec{u}_1$ and $\vec{r} = \vec{a}_2 + \mu\,\vec{u}_2$ is given by

$$d = \left|\frac{(\vec{u}_1 \times \vec{u}_2).(\vec{a}_2 - \vec{a}_1)}{|\vec{u}_1 \times \vec{u}_2|}\right|$$



REMARK

The shortest distance between two skew lines

$d =$ Projection of $(\vec{a}_2 - \vec{a}_1)$ along $(\vec{u}_1 \times \vec{u}_2)$

$= |\,(\vec{a}_2 - \vec{a}_1).[\text{unit Vector along } (\vec{u}_1 \times \vec{u}_2)]\,|$ $\qquad \left(\text{using } \hat{a} = \dfrac{\vec{a}}{|\vec{a}|}\right)$

$$= \left|\frac{(\vec{a}_2 - \vec{a}_1).(\vec{u}_1 \times \vec{u}_2)}{|\vec{u}_1 \times \vec{u}_2|}\right|$$

7.2 CARTESIAN FORM

The shortest distance between the lines l_1 and l_2 whose cartesian equations are

$$\frac{x-x_1}{a_1}=\frac{y-y_1}{b_1}=\frac{z-z_1}{c_1} \text{ and } \frac{x-x_2}{a_2}=\frac{y-y_2}{b_2}=\frac{z-z_2}{c_2} \text{ is given by}$$

$$d=\left|\frac{\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}}{\sqrt{(b_1c_2-b_2c_1)^2+(c_1a_2-c_2a_1)^2+(a_1b_2-a_2b_1)^2}}\right|$$

REMARKS

(i) For the question "Show that the following lines intersect. Also find the point of intersection", solve by using "general points on the lines".

(ii) For the question "Check if the following lines intersect. ", solve by using "concept of Shortest Distance".

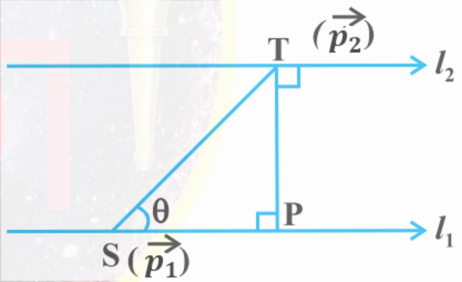
(iii) Be clear with the meaning of '**hence find** . . . " and "**hence or otherwise find** . . ."

8. DISTANCE BETWEEN TWO PARALLEL LINES

Distance Between two parallel lines l_1 and l_2 whose vector equations are

$\vec{r}=\vec{p}_1+\lambda\vec{u}$ and $\vec{r}=\vec{p}_2+\lambda\vec{u}$ is given by

$$d=\left|\frac{\vec{u}\times(\vec{p}_2-\vec{p}_1)}{|\vec{u}|}\right|$$



IMPORTANT REMARKS

(i) To find the distance between two parallel lines, first write the equations in such a form that the vectors attached with Type equation here.

For Example,

To find the distance between the parallel lines

$\vec{r}=\hat{\imath}+2\hat{\jmath}+3\hat{k}+\lambda(\hat{\imath}-3\hat{\jmath}+2\hat{k})$ and $\vec{r}=3\hat{\imath}-2\hat{\jmath}+4\hat{k}+\mu(-2\hat{\imath}+6\hat{\jmath}-4\hat{k})$

first write the first equation as

$\vec{r}=3\hat{\imath}-2\hat{\jmath}+4\hat{k}+(-2)\lambda(\hat{\imath}-3\hat{\jmath}+2\hat{k})=\vec{r}=3\hat{\imath}-2\hat{\jmath}+4\hat{k}+\lambda'(\hat{\imath}-3\hat{\jmath}+2\hat{k})$ where $\lambda'=(-2)\lambda$ and then take $\vec{u}=\hat{\imath}-3\hat{\jmath}+2\hat{k}$ in the formula.