FORMULAE & KEY POINTS
CLASS 12 MATHEMATICS
CHAPTER 07 : INTEGRALS

PART I : INDEFINITE INTEGRALS

PART II : DEFINITE INTEGRALS

PART I : INDEFINITE INTEGRALS

1.1 ANTI DERIVATIVES (or PRIMITIVE), INDEFINITE INTEGRAL AND INTEGRATION

Let f be a function which is differentiable in an interval I, i.e., its derivative f' exists at each point of I, then the functions that could possibly have given function as a derivative are called **ANTI DERIVATIVES (or PRIMITIVE)** of the function.
Further, the formula that gives all these anti derivatives is called the **INDEFINITE INTEGRAL** of the function and such process of finding anti derivatives is called **INTEGRATION or ANTI DIFFERENTIATION**.
Thus, integration is the inverse process of differentiation.

1.2 INDEFINITE INTEGRALS, DEFINITE INTEGRALS AND INTEGRAL CALCULUS

The development of integral calculus arises out of the efforts of solving the problems of the following types:
(a) the problem of finding a function whenever its derivative is given,
(b) the problem of finding the area bounded by the graph of a function under certain conditions.
These two problems lead to the two forms of the integrals, e.g., **INDEFINITE INTEGRALS** and **DEFINITE INTEGRALS**, which together constitute the **INTEGRAL CALCULUS**

REMARK:

(i) There is a connection, known as the **FUNDAMENTAL THEOREM OF CALCULUS**, between indefinite integral and definite integral which makes the definite integral as a practical tool for science and engineering.

(ii) Here, we are given the derivative of a function and asked to find its primitive, i.e., the original function. Such a process is called integration or anti differentiation.

(iii) If there is a function F such that $\frac{d}{dx}\mathrm{F}(x) = f(x)\ \forall\ x \in \mathrm{I}$, then for any arbitrary real number C (called CONSTANT OF INTEGRATION,

$$\frac{d}{dx}[\mathrm{F}(x) + \mathrm{C}] = f(x), x \in \mathrm{I}$$

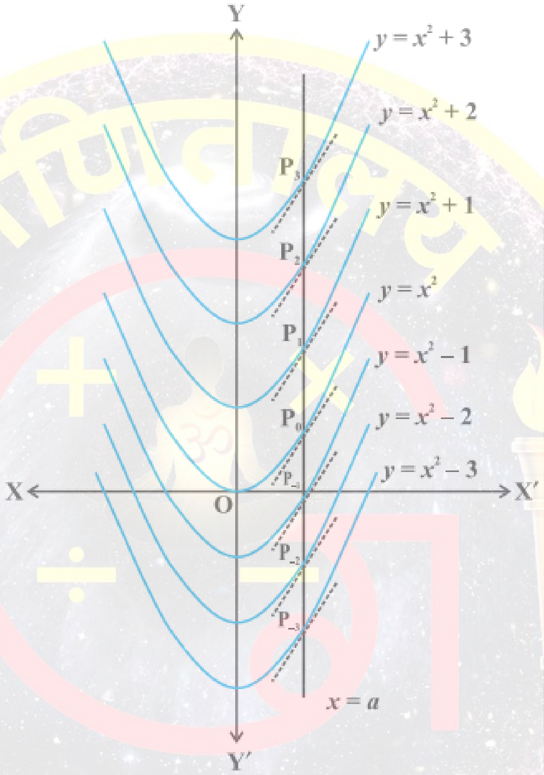
Thus, $\{\mathrm{F} + \mathrm{C}, \mathrm{C} \in \mathrm{R}\}$ denotes a family of anti derivatives of f

(iv) We introduce a new symbol, namely, $\int f(x)\,dx$ (read as "***the indefinite integral of f with respect to x***") which represents the entire class of anti derivatives.

Symbolically, we write $\int f(x)\,dx = F(x) + C$.

(v) In practice, we normally do not mention the interval over which the various functions are defined. However, in any specific problem one has to keep it in mind.

1.3 GEOMETRICAL INTERPRETATION OF INDEFINITE INTEGRAL



The expression $\int f(x)\,dx = F(x) + C$, represents a family of curves with the following properties:

(i) The different values of C will correspond to different members of this family

(ii) The above members can be obtained by shifting any one of the curves parallel to itself.

(iii) If we draw a line parallel to y-axis, say $x = a$, then the tangents at all the points where this line intersects the family of curve $F(x) + C$ are parallel to each other.

1.4 Symbols/Terms/Phrases Meaning

	Symbols/Terms/Phrases	Meaning
(i)	$\int f(x)dx$	Integral of f with respect to x
(ii)	$f(x)$ in $\int f(x)dx$	Integrand
(iii)	x in $\int f(x)dx$	Variable of integration
(iv)	Integrate	Find the integral
(v)	An integral of f	A function F such that $F'(x) = f(x)$

(vi)	Integration	The process of finding the integral
(vii)	Constant of Integration	Any real number C, considered as constant function

2. Some Properties of Indefinite Integrals

(i) $\frac{d}{dx}\int f(x)\,dx = f(x)$

(ii) $\int f'(x)\,dx = f(x) + C$

(iii) $\int [f(x) \pm g(x)]\,dx = \int f(x)\,dx \pm \int g(x)\,dx$

(iv) $\int k.f(x)\,dx = k\int f(x)\,dx$, where k is a real number.

(v) $\int f(x) \times g(x)\,dx \neq \int f(x)\,dx \times \int g(x)\,dx$

1. Integrals of Algebraic Functions – I

(i) $\int x^n dx = \frac{x^{n+1}}{n+1} + C, \quad n \neq -1$

(ii) $\int \frac{1}{x} dx = \log|x| + C$

(iii) $\int x\,dx = \frac{x^2}{2} + C$

(iv) $\int dx = x + C$

(v) $\int \sqrt{x}\,dx = \frac{x^{3/2}}{3/2} + C = \frac{2}{3}x^{3/2} + C$

(vi) $\int \frac{1}{\sqrt{x}} dx = 2\sqrt{x} + C$

(vii) $\int \frac{1}{x^n} dx = \frac{1}{(-n+1)x^{n-1}} + C, \quad n \neq 1$

Example: $\int \frac{1}{x^5} dx = \frac{1}{-4x^4} + C$

2. Integrals of Algebraic Functions – II

(i) $\int (ax+b)^n dx = \frac{(ax+b)^{n+1}}{(n+1).a} + C, \quad n \neq -1$

(ii) $\int \frac{1}{(ax+b)} dx = \log|ax+b| + C$

(iii) $\displaystyle\int \frac{1}{(ax+b)^n}dx = \frac{1}{(-n+1)(ax+b)^{n-1}.a} + C, \ \ n \neq 1$

Example: $\displaystyle\int \frac{1}{(3x-7)^5}dx = \frac{1}{-4(3x-7)^4.3} + C$

3. Some Special Types of Integrals:

(i) $\displaystyle\int [f(x)]^n . f'(x)dx = \frac{[f(x)]^{n+1}}{n+1} + C, \ \ n \neq -1$

(ii) $\displaystyle\int \frac{f'(x)}{f(x)}dx = \log|f(x)| + C$

(iii) $\displaystyle\int \frac{f'(x)}{[f(x)]^n}dx = \frac{1}{(-n+1)[f(x)]^{n-1}} + C, \ \ n \neq 1$

4. Integrals of Trigonometric Functions:

(i) $\displaystyle\int \sin x \, dx = \cos x + C$

(ii) $\displaystyle\int \cos x \, dx = -\sin x + C$

(iii) $\displaystyle\int \tan dx = -\log|\cos x| + C = \log|\sec x| + C$

(iv) $\displaystyle\int \cot dx = \log|\sin x| + C = -\log|\operatorname{cosec} x| + C$

(v) $\displaystyle\int \sec x \, dx = \log|\sec x + \tan x| + C = \log\left|\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)\right| + C$

(vi) $\displaystyle\int \operatorname{cosec} x \, dx = \log|\operatorname{cosec} x - \cot x| + C = \log\left|\tan\frac{x}{2}\right| + C$

5. Integrals of Involving Inverse Trigonometric Functions:

(i) $\displaystyle\int \frac{1}{\sqrt{1-x^2}}dx = \sin^{-1}x + C = -\cos^{-1}x + C$

(ii) $\displaystyle\int \frac{-1}{\sqrt{1-x^2}}dx = \cos^{-1}x + C = -\sin^{-1}x + C$

(iii) $\displaystyle\int \frac{1}{1+x^2}dx = \tan^{-1}x + C = -\cot^{-1}x + C$

(iv) $\displaystyle\int \frac{-1}{1+x^2}dx = \cot^{-1}x + C = -\tan^{-1}x + C$

(v) $\displaystyle\int \frac{1}{x\sqrt{x^2-1}}dx = \sec^{-1}x + C = -\operatorname{cosec}^{-1}x + C$

(vi) $\displaystyle\int \frac{-1}{x\sqrt{x^2-1}}dx = \operatorname{cosec}^{-1}x + C = -\sec^{-1}x + C$

6. Integrals of Exponential Functions:

(i) $\int a^x dx = \frac{a^x}{\log a} + C$

(ii) $\int e^x dx = e^x + C$

7. Some Special Integral Forms:

(i) $\int \frac{e^x + e^{-x}}{e^x - e^{-x}} dx = \log |e^x - e^{-x}| + C$

(ii) $\int \frac{e^x - e^{-x}}{e^x + e^{-x}} dx = \log |e^x + e^{-x}| + C$

(iii) $\int \frac{\sin x + \cos x}{\sin x - \cos x} dx = \log |\sin x - \cos x| + C$

(iv) $\int \frac{\sin x - \cos x}{\sin x + \cos x} dx = -\log |\sin x + \cos x| + C$

(v) To find $\int \frac{dx}{e^x + 1}$ multiply numerator and denominator by e^x as $\int \frac{e^x . dx}{e^x . (e^x + 1)}$ and put $e^x = t$

(vi) To find $\int \frac{dx}{x(1 + x^n)}$ write it as $\int \frac{dx}{x . x^n \left(1 + \frac{1}{x^n}\right)} = \int \frac{dx}{x^{n+1} \left(1 + \frac{1}{x^n}\right)}$ and put $1 + \frac{1}{x^n} = t$

8.1 Integrals of Some Particular Functions:

(i) $\int \frac{dx}{x^2 - a^2} = \frac{1}{2a} \log \left|\frac{x - a}{x + a}\right| + C$

(ii) $\int \frac{dx}{a^2 - x^2} = \frac{1}{2a} \log \left|\frac{a + x}{a - x}\right| + C$

(iii) $\int \frac{dx}{x^2 + a^2} = \frac{1}{a} \tan^{-1} \frac{x}{a} + C$

8.2 Integrals of Some Particular Functions:

(i) $\int \frac{dx}{\sqrt{x^2 - a^2}} = \log \left|x + \sqrt{x^2 - a^2}\right| + C$

(ii) $\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1} \frac{x}{a} + C$

(iii) $\int \frac{dx}{\sqrt{x^2 + a^2}} = \log \left|x + \sqrt{x^2 + a^2}\right| + C$

8.3 Integrals of Some Particular Functions:

(i) $\int \sqrt{x^2 - a^2}\, dx = \frac{x}{2} \sqrt{x^2 - a^2} - \frac{a^2}{2} \log \left|x + \sqrt{x^2 - a^2}\right| + C$

(ii) $\int \sqrt{a^2 - x^2}\, dx = \frac{x}{2} \sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1} \frac{x}{a} + C$

(iii) $\int \sqrt{x^2+a^2}\,dx = \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\log\left|x+\sqrt{x^2+a^2}\right| + C$

9. Integrals of Rational Function $\frac{P(x)}{D(x)}$ by the Method of Partial Fractions:

9.1 Most Important:

First of all check whether Degree of Numerator < Degree of Denominator.
If Degree of Numerator $\geq$ Degree of Denominator, divide the Numerator by Denominator and express it in the form $R(x) + \frac{P(x)}{Q(x)}$ and then proceed further

Example: In $\int \frac{x^2+1}{x^2-5x+6}dx$, Deg N^r = Deg $D^r = 2$, $\therefore$ we divide N^r by D^r and write it as $\int \left[1 + \frac{5x-5}{x^2-5x+6}\right]$ and then proceed further

9.2 Form of the Rational Function — Form of the Partial Fraction

	Form of the Rational Function	Form of the Partial Fraction
(i)	$\frac{px+q}{(x-a)(x-b)}$, $a \neq b$	$\frac{A}{(x-a)} + \frac{B}{(x-b)}$
(ii)	$\frac{px+q}{(x-a)^2}$	$\frac{A}{(x-a)} + \frac{B}{(x-b)^2}$
(iii)	$\frac{px^2+qx+r}{(x-a)(x-b)(x-c)}$	$\frac{A}{(x-a)} + \frac{B}{(x-b)} + \frac{C}{(x-c)}$
(iv)	$\frac{px^2+qx+r}{(x-a)(x^2+bx+c)}$	$\frac{A}{(x-a)} + \frac{Bx+C}{(x^2+bx+c)}$

where x^2+bx+c cannot be factorised further
In the above table, A, B and C are real numbers to be determined suitably

Remark:
If all the powers of x in the rational function are even, the we apply the method of partial fractions as given in the following example:

Example : Find $\int \frac{x^2}{(x^2+1)(x^2+4)}dx$

Solution : Put $x^2 = y$ and take

$$\frac{x^2}{(x^2+1)(x^2+4)} = \frac{y}{(y+1)(y+4)} = \frac{A}{y+1} + \frac{B}{y+4} \Rightarrow A = -\frac{1}{3}, B = \frac{4}{3}$$

Thus, $$\frac{x^2}{(x^2+1)(x^2+4)} = -\frac{1}{3(x^2+1)} + \frac{4}{3(x^2+4)}$$

$$\therefore \int \frac{x^2}{(x^2+1)(x^2+4)}dx = -\frac{1}{3}\int \frac{1}{(x^2+1)}dx + \frac{4}{3}\int \frac{1}{(x^2+4)}dx$$

10. Integration by Parts OR Product Rule of Integration:

10.1 $\int (I) \times (II)\,dx = (I) \times \int (II)dx - \int \left[\frac{d}{dx}(I) \times \int (II)\,dx\right]dx$

10.2 "The integral of the product of two functions = (first function) × (integral of the second function) − Integral of [(differential coefficient of the first function) × (integral of the second function)]"

Remarks:

(i) Integration by parts is not applicable to product of functions in all cases. For instance, the method does not work for $\int \sqrt{x} \sin x \, dx$.

The reason is that there does not exist any function whose derivative is $\sqrt{x} \sin x$

(ii) While finding the integral of the second function, we do not add any constant of integration because adding a constant to the integral of the second function is superfluous. so far as the final result is concerned.

(iii) While applying the method of integration by parts, the order of preference of the Fisrt Function and the Second Function is taken as follows:

I − Inverse Trigonometric Function

L − Logarithmic Function

A − Algebraic Function

T − Trigonometric Function

E − Exponential Function

In case both the functions are of same type, we take the First Function as that whose integration is complicated.

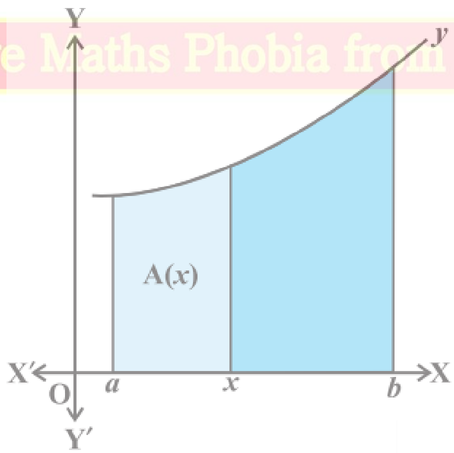
(iv) To integrate the function of the type $\int \log x \, dx$ and $\int \sin^{-1} x \, dx$ we write these as $\int (\log x).1 \, dx$ and $\int (\sin^{-1} x).1 \, dx$ and apply integration by parts by taking 1 as the second function.

(v) $\int e^x [\, f(x) + f'(x) \,] \, dx = e^x f(x) + C$ **(VERY IMPORTANT)**

Example : Find $\int e^x [\sin x + \cos x \,] \, dx = e^x \sin x + C$

PART II : DEFINITE INTEGRALS

1. Area function



Let f be a function continuos in the closed interval $[\, a, b \,]$ Then, the definite integral

$\int_a^b f(x)dx$ represents the area of the region bounded by the curve $y = f(x), a \le x \le b$, the $x-$axis and the ordinates $x = a$ and $x = b$.

Further, let x be a given point in $[a, b]$. Then A $(x) = \int_a^x f(x)dx$ represents the area function.

2. First Fundamental Theorem of Integral Calculus

Let the area function be defined by A $(x) = \int_a^x f(x)dx$ for all $x \ge a$, where the function f is continuous on [a, b]. Then

$$A'(x) = f(x) \text{ for all } x \in [a, b].$$

3. Second Fundamental Theorem of Integral Calculus

Let f be a continuous function of x defined on the closed interval [a, b]and let F be another function such that $\frac{d}{dx}F(x) = f(x)$ for all x in the domain of f, then $\int_a^b f(x)dx = F(b) - F(a)$

This is called the definite integral of f over the range [a, b], where a and b are called the limits of integration, a being the lower limit and b the upper limit.

REMARK

In $\int_a^b f(x)dx$ the function f needs to be well defined and continuous in [a, b]. For instance, the consideration of definite integral $\int_{-2}^3 x\,(x^2-1)^{1/2}\,dx$ is erroneous since the function f expressed by $f(x) = x\,(x^2-1)^{1/2}$ is not defined in a portion $-1 < x < 1$ of the closed interval [– 2, 3].

4. PROPERTIES OF DEFINITE INTEGRALS:

P_0 $\int_a^b f(x)dx = \int_a^b f(t)dt$ (Since the result of the definite integral does not contain any variable. Due to this reason, the variable of definite integral is called a 'Dummy Variable'.

P_1 $\int_a^b f(x)dx = -\int_b^a f(x)dx$. In particular, $\int_a^a f(x)dx = 0$

Example

$$\int_{\frac{\pi}{2}}^0 -\cos x\,dt = \int_0^{\frac{\pi}{2}} \cos x\,dt = [\sin x]_0^{\frac{\pi}{2}} = \sin\frac{\pi}{2} - \sin 0 = 1 - 0 = 1$$

P_2 $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$, where $a < c < b$

Example

Consider $\mathrm{I} = \int_{-5}^{5} |x+2|\, dx$

As $|x+2| = \begin{cases} (x+2) \text{ if } x \geq -2 \\ -(x+2) \text{ if } x < -2 \end{cases}$

$\therefore \mathrm{I} = \int_{-5}^{-2} -(x+2)dx + \int_{-2}^{5} (x+2)dx$ . . . and so on.

P3 $\int_a^b f(x)dx = \int_a^c f(a+b-x)dx$

Example

$$\int_{\pi/6}^{\pi/3} \frac{dx}{1+\sqrt{\tan x}} = \int_{\pi/6}^{\pi/3} \frac{\sqrt{\cos}\ dx}{\sqrt{\cos x} + \sqrt{\sin x}} = \int_{\pi/6}^{\pi/3} \frac{\sqrt{\cos\left(\frac{\pi}{6}+\frac{\pi}{3}-x\right)}\ dx}{\sqrt{\cos\left(\frac{\pi}{6}+\frac{\pi}{3}-x\right)} + \sqrt{\sin\left(\frac{\pi}{6}+\frac{\pi}{3}-x\right)}}$$

$$= \int_{\pi/6}^{\pi/3} \frac{\sqrt{\cos\left(\frac{\pi}{2}-x\right)}\ dx}{\sqrt{\cos\left(\frac{\pi}{2}-x\right)} + \sqrt{\sin\left(\frac{\pi}{2}-x\right)}} = \int_{\pi/6}^{\pi/3} \frac{\sqrt{\sin}\ dx}{\sqrt{\sin x} + \sqrt{\cos x}}$$ . . . and so on.

P4 $\int_a^b f(x)dx = \int_a^c f(a-x)dx$

REMARK:

This is a particular case of P_3, which can be obtained by taking $a = 0$ and then $b = a$.

Example

$$\int_0^{\frac{\pi}{2}} \frac{\sin^{\frac{3}{2}} x}{\sin^{\frac{3}{2}} x + \cos^{\frac{3}{2}} x} dx = \int_0^{\frac{\pi}{2}} \frac{\sin^{\frac{3}{2}} x \left(\frac{\pi}{2}-x\right)}{\sin^{\frac{3}{2}}\left(\frac{\pi}{2}-x\right) + \cos^{\frac{3}{2}}\left(\frac{\pi}{2}-x\right)} dx$$

$$= \int_0^{\pi/2} \frac{\cos^{\frac{3}{2}} x}{\cos^{\frac{3}{2}} x + \sin^{\frac{3}{2}} x} dx$$ . . . and so on.

P5 $\int_0^{2a} f(x)dx = \int_0^a f(x)dx + \int_0^a f(2a-x)dx$

P6 $\int_0^{2a} f(x)dx = \begin{cases} 2\int_0^a f(x)dx \text{ if } f(2a-x) = f(x) \text{ and} \\ 0 \text{ if } f(2a-x) = -f(x) \end{cases}$

Example

Consider, $\mathrm{I} = \int_0^{2\pi} \cos^5 x\ dx$

As $f(2\pi - x) = \cos(2\pi - x) = -\cos x = -f(x)$

$\therefore \mathrm{I} = 0$

Example

Consider, $I = \int_0^{\pi} \log \sin x \, dx$

As $f(\pi - x) = \log \sin(\pi - x) = \log \sin x = f(x)$

$\therefore I = 2\int_0^{\frac{\pi}{2}} \log \sin x \, dx. . .$ and so on

P7 $\int_{-a}^{a} f(x)dx = \begin{cases} 2\int_0^a f(x)dx & f \text{ is an } \textbf{Even Function} \text{ i.e. if } f(-x) = f(x) \\ 0 & \text{if is an } \textbf{Odd Function} \text{ i.e. if } f(-x) = -f(x) \end{cases}$

Example

Consider $I = \int_{-1}^{1} \frac{x^3 + |x| + 1}{x^2 + 2|x| + 1} dx$

$\therefore I = \int_{-1}^{1} \frac{x^3}{x^2 + 2|x| + 1} dx + \int_{-1}^{1} \frac{|x| + 1}{x^2 + 2|x| + 1} dx = I_1 + I_2 \text{ (let)} dx$

Now, consider $I_1 = \int_{-1}^{1} \frac{x^3}{x^2 + 2|x| + 1} dx$. Here, $f(-x) = \frac{(-x)^3}{(-x)^2 + 2|-x| + 1} =$

$= \frac{-x^3}{x^2 + 2|x| + 1} = -f(x)$ i.e. f is an odd function.

$\therefore I_1 = 0$

Next, consider $I_2 = \int_{-1}^{1} \frac{|x| + 1}{x^2 + 2|x| + 1} dx$. Here, $g(-x) = \frac{|-x| + 1}{(-x)^2 + 2|-x| + 1}$

$= \frac{|x| + 1}{x^2 + 2|x| + 1} = g(x)$

$\therefore I_2 = 2\int_0^1 \frac{|x| + 1}{x^2 + 2|x| + 1} dx$

$= 2\int_0^1 \frac{x + 1}{x^2 + 2x + 1} dx$ (as $|x| = x$ for $0 \le x \le 1$) . . . and so on.